Page 166, equation (4.89)

Equation (4.89) can be obtained from (4.87) relatively easily. Here we present a longer derivation based on (4.82) which also hides some expressions necessary for the procedure starting with (4.87).

We will start with modification of some terms appearing in (4.84).

$$\mathrm{div}(\mathbf{q}/T) = (1/T)\mathrm{div}\mathbf{q} + \mathbf{q}.\mathrm{grad}(1/T). \tag{1}$$

From (4.82) we have:

$$\mathrm{div}\mathbf{q} - Q = \sum_{\alpha=1}^{n} \mathrm{tr}\,\mathbf{T}_\alpha \mathbf{D}_\alpha - \sum_{\beta=1}^{n-1} \mathbf{k}_\beta.\mathbf{u}_\beta - \frac{1}{2}\sum_{\beta=1}^{n-1} r_\beta \mathbf{u}_\beta^2 - \sum_{\alpha=1}^{n} \frac{\partial \rho_\alpha u_\alpha}{\partial t} - \sum_{\alpha=1}^{n} \mathrm{div}(\rho_\alpha u_\alpha \mathbf{v}_\alpha). \tag{2}$$

Note that the stretching can be written in the form of (4.88); then

$$\mathrm{tr}\,\mathbf{T}_\alpha \mathbf{D}_\alpha = \mathrm{tr}\,\mathbf{T}_\alpha (\overset{\circ}{\mathbf{D}}_\alpha + (1/3)\mathrm{tr}\,\mathbf{D}_\alpha \mathbf{1}) = \mathrm{tr}\,\mathbf{T}_\alpha \overset{\circ}{\mathbf{D}}_\alpha + (1/3)\mathrm{tr}\,\mathbf{T}_\alpha \mathrm{tr}\,\mathbf{D}_\alpha. \tag{3}$$

The time derivative in (2) can be expanded after introducing the free energy

$$-\frac{\partial \rho_\alpha u_\alpha}{\partial t} = -\frac{\partial \rho_\alpha (f_\alpha + T s_\alpha)}{\partial t} = -\frac{\partial \rho_\alpha f_\alpha}{\partial t} - \frac{\partial \rho_\alpha T s_\alpha}{\partial t}. \tag{4}$$

Similarly, the divergence term in (2) can be modified:

$$\mathrm{div}(\rho_\alpha u_\alpha \mathbf{v}_\alpha) = \mathrm{div}\rho_\alpha (f_\alpha + T s_\alpha)\mathbf{v}_\alpha + \rho_\alpha (f_\alpha + T s_\alpha)\mathrm{div}\mathbf{v}_\alpha + \mathbf{v}_\alpha.\mathrm{grad}\rho_\alpha (f_\alpha + T s_\alpha). \tag{5}$$

Substituting from (1)-(5) into (4.84) it follows:

$$\begin{aligned} T\sigma &= T\sum_{\alpha=1}^{n} \frac{\partial \rho_\alpha s_\alpha}{\partial t} + T\sum_{\alpha=1}^{n} \mathrm{div}(\rho_\alpha s_\alpha \mathbf{v}_\alpha) + \mathrm{div}\mathbf{q} + T\mathbf{q}.\mathrm{grad}(1/T) - Q \\ &= T\sum_{\alpha=1}^{n} \frac{\partial \rho_\alpha s_\alpha}{\partial t} + T\sum_{\alpha=1}^{n} \mathrm{div}(\rho_\alpha s_\alpha \mathbf{v}_\alpha) + \sum_{\alpha=1}^{n} \mathrm{tr}\,\mathbf{T}_\alpha \overset{\circ}{\mathbf{D}}_\alpha \\ &+ (1/3)\sum_{\alpha=1}^{n} \mathrm{tr}\,\mathbf{T}_\alpha \mathrm{tr}\,\mathbf{D}_\alpha - \sum_{\beta=1}^{n-1} \mathbf{k}_\beta.\mathbf{u}_\beta - \frac{1}{2}\sum_{\beta=1}^{n-1} r_\beta \mathbf{u}_\beta^2 - \sum_{\alpha=1}^{n} \frac{\partial \rho_\alpha f_\alpha}{\partial t} \\ &- \sum_{\alpha=1}^{n} \frac{\partial \rho_\alpha T s_\alpha}{\partial t} - \sum_{\alpha=1}^{n} \rho_\alpha (f_\alpha + T s_\alpha)\mathrm{div}\mathbf{v}_\alpha - \sum_{\alpha=1}^{n} \mathbf{v}_\alpha.\mathrm{grad}\rho_\alpha (f_\alpha + T s_\alpha) \\ &+ T\mathbf{q}.\mathrm{grad}(1/T). \end{aligned} \tag{6}$$

Several modifications and simplifications can be made in (6):

$$T\sum_{\alpha=1}^{n}\frac{\partial\rho_\alpha s_\alpha}{\partial t}-\sum_{\alpha=1}^{n}\frac{\partial\rho_\alpha T s_\alpha}{\partial t}=-\sum_{\alpha=1}^{n}\rho_\alpha s_\alpha\frac{\partial T}{\partial t}; \tag{7}$$

$$T\mathbf{q}.\mathrm{grad}(1/T)=T\mathbf{q}.\left[-(1/T^2)\mathrm{grad}T\right]=-(1/T)\mathbf{q}.\mathrm{grad}T; \tag{8}$$

$$T\sum_{\alpha=1}^{n}\mathrm{div}(\rho_\alpha s_\alpha\mathbf{v}_\alpha)=T\sum_{\alpha=1}^{n}\left[\rho_\alpha s_\alpha\mathrm{div}\mathbf{v}_\alpha+\mathbf{v}_\alpha.\mathrm{grad}(\rho_\alpha s_\alpha)\right]; \tag{9}$$

$$-\sum_{\alpha=1}^{n}\rho_\alpha(f_\alpha+Ts_\alpha)\mathrm{div}\mathbf{v}_\alpha=-\sum_{\alpha=1}^{n}\rho_\alpha f_\alpha\mathrm{div}\mathbf{v}_\alpha-T\sum_{\alpha=1}^{n}\rho_\alpha s_\alpha\mathrm{div}\mathbf{v}_\alpha; \tag{10}$$

$$\begin{aligned}-\sum_{\alpha=1}^{n}\mathbf{v}_\alpha.\mathrm{grad}\rho_\alpha(f_\alpha+Ts_\alpha)&=-\sum_{\alpha=1}^{n}\mathbf{v}_\alpha.\mathrm{grad}(\rho_\alpha f_\alpha)-\sum_{\alpha=1}^{n}\mathbf{v}_\alpha.\mathrm{grad}(\rho_\alpha T s_\alpha)=\\&-\sum_{\alpha=1}^{n}\mathbf{v}_\alpha.\mathrm{grad}(\rho_\alpha f_\alpha)-\sum_{\alpha=1}^{n}\mathbf{v}_\alpha.T\mathrm{grad}(\rho_\alpha s_\alpha)\\&-\sum_{\alpha=1}^{n}\mathbf{v}_\alpha.\rho_\alpha s_\alpha\mathrm{grad}T.\end{aligned} \tag{11}$$

The first term in r.h.s. of (9) and the second term in r.h.s. of (10) cancel; because $\mathrm{div}\mathbf{v}_\alpha=\mathrm{tr}(\mathrm{grad}\mathbf{v}_\alpha)=\mathrm{tr}\,\mathbf{D}_\alpha$ (remember that $\mathbf{W}_\alpha$ is skew-symmetric) only the term $-\sum_{\alpha=1}^{n}\rho_\alpha f_\alpha\mathrm{tr}\,\mathbf{D}_\alpha$ finally remains from the combination of (9) and (10). The second term in r.h.s. of (9) and the second term in r.h.s. of (11) cancel. The remaining first term of r.h.s. of (11) can be written as $-\sum_{\beta=1}^{n-1}\mathbf{u}_\beta.\mathrm{grad}(\rho_\beta f_\beta)-\mathbf{v}_n.\sum_{\alpha=1}^{n}\mathrm{grad}(\rho_\alpha f_\alpha)$. Similarly, the third term of r.h.s. of (11): $-\sum_{\beta=1}^{n-1}\rho_\beta s_\beta\mathbf{u}_\beta.\mathrm{grad}T-\sum_{\alpha=1}^{n}\rho_\alpha s_\alpha\mathbf{v}_n.\mathrm{grad}T$.

Introducing all the above derivations and considerations into (6) gives following final expression:

$$\begin{aligned}-T\sigma=&\sum_{\alpha=1}^{n}\rho_\alpha s_\alpha\frac{\partial T}{\partial t}-\sum_{\alpha=1}^{n}\mathrm{tr}\,\mathbf{T}_\alpha\overset{\circ}{\mathbf{D}}_\alpha-(1/3)\sum_{\alpha=1}^{n}\mathrm{tr}\,\mathbf{T}_\alpha\mathrm{tr}\,\mathbf{D}_\alpha+\sum_{\beta=1}^{n-1}\mathbf{k}_\beta.\mathbf{u}_\beta\\&+\sum_{\alpha=1}^{n}\frac{\partial\rho_\alpha f_\alpha}{\partial t}+\frac{1}{2}\sum_{\beta=1}^{n-1}r_\beta\mathbf{u}_\beta^2+\sum_{\alpha=1}^{n}\rho_\alpha f_\alpha\mathrm{tr}\mathbf{D}_\alpha+\sum_{\beta=1}^{n-1}\mathbf{u}_\beta.\mathrm{grad}(\rho_\beta f_\beta)\\&+\mathbf{v}_n.\mathrm{grad}\sum_{\alpha=1}^{n}\rho_\alpha f_\alpha+\sum_{\beta=1}^{n-1}\rho_\beta s_\beta\mathbf{u}_\beta.\mathrm{grad}T+\mathbf{v}_n.\mathrm{grad}T\sum_{\alpha=1}^{n}\rho_\alpha s_\alpha\\&+(1/T)\mathbf{q}.\mathrm{grad}T.\end{aligned} \tag{12}$$

Eq. (12) is (4.89).