Page 164, equation (4.82)

Multiplying (4.56) with $\mathbf{v}_\alpha$ results in:

$$\rho_\alpha \dot{\mathbf{v}}_\alpha.\mathbf{v}_\alpha \equiv \rho_\alpha \frac{1}{2}\dot{\overline{\mathbf{v}_\alpha^2}} = \mathbf{v}_\alpha.\mathrm{div}\,\mathbf{T}_\alpha + \rho_\alpha \mathbf{b}_\alpha.\mathbf{v}_\alpha + \mathbf{k}_\alpha.\mathbf{v}_\alpha. \tag{1}$$

Summing up (1) for all components and subtracting the result from (4.81) we obtain:

$$\sum_{\alpha=1}^{n} \rho_\alpha \dot{u}_\alpha + \sum_{\alpha=1}^{n} r_\alpha \mathbf{u}_\alpha + \sum_{\alpha=1}^{n} r_\alpha \frac{1}{2}\mathbf{v}_\alpha^2 = \mathrm{div} \sum_{\alpha=1}^{n} \mathbf{v}_\alpha \mathbf{T}_\alpha - \sum_{\alpha=1}^{n} \mathbf{v}_\alpha.\mathrm{div}\,\mathbf{T}_\alpha - \mathrm{div}\mathbf{q} + Q - \sum_{\alpha=1}^{n} \mathbf{k}_\alpha.\mathbf{v}_\alpha. \tag{2}$$

The divergence $\mathrm{div}\,\mathbf{v}_\alpha \mathbf{T}_\alpha$ can be modified on the same basis as in the case of (3.98), see eqs. (3) and (5) in the derivation of it, and the symmetry of $\mathbf{T}_\alpha$, (4.70):

$$\begin{aligned}\mathrm{div}\,\mathbf{v}_\alpha \mathbf{T}_\alpha &= \mathrm{tr}\,\mathbf{L}_\alpha \mathbf{T}_\alpha + \mathbf{v}_\alpha.\mathrm{div}\,\mathbf{T}_\alpha = \mathrm{tr}\,\mathbf{D}_\alpha \mathbf{T}_\alpha + \mathbf{v}_\alpha.\mathrm{div}\,\mathbf{T}_\alpha \\ &= \mathrm{tr}\,\mathbf{T}_\alpha \mathbf{D}_\alpha + \mathbf{v}_\alpha.\mathrm{div}\,\mathbf{T}_\alpha\end{aligned} \tag{3}$$

(for verification of the trace equivalence see at the end).

Further, following sum is modified using the definition of diffusion velocity (4.24):

$$\begin{aligned}\sum_{\alpha=1}^{n} \mathbf{k}_\alpha.\mathbf{v}_\alpha + \frac{1}{2}\sum_{\alpha=1}^{n} r_\alpha \mathbf{v}_\alpha^2 &= \sum_{\alpha=1}^{n} \mathbf{k}_\alpha.(\mathbf{u}_\alpha + \mathbf{v}_n) + \frac{1}{2}\sum_{\alpha=1}^{n} r_\alpha (\mathbf{u}_\alpha + \mathbf{v}_n)^2 = \\ &\sum_{\alpha=1}^{n} \mathbf{k}_\alpha.\mathbf{u}_\alpha + (\sum_{\alpha=1}^{n} \mathbf{k}_\alpha).\mathbf{v}_n + \frac{1}{2}\sum_{\alpha=1}^{n} r_\alpha \mathbf{u}_\alpha^2 + \\ &\sum_{\alpha=1}^{n} r_\alpha \mathbf{u}_\alpha.\mathbf{v}_n + \frac{1}{2}\sum_{\alpha=1}^{n} r_\alpha \mathbf{v}_n^2\end{aligned}$$

and then combined with (4.20) and (4.63):

$$\begin{aligned}\sum_{\alpha=1}^{n} \mathbf{k}_\alpha.\mathbf{v}_\alpha + \frac{1}{2}\sum_{\alpha=1}^{n} r_\alpha \mathbf{v}_\alpha^2 = \sum_{\alpha=1}^{n} \mathbf{k}_\alpha.\mathbf{u}_\alpha - (\sum_{\alpha=1}^{n} r_\alpha \mathbf{v}_\alpha).\mathbf{v}_n \\ + \frac{1}{2}\sum_{\alpha=1}^{n} r_\alpha \mathbf{u}_\alpha + (\sum_{\alpha=1}^{n} r_\alpha \mathbf{u}_\alpha).\mathbf{v}_n.\end{aligned} \tag{4}$$

The sum of the second and fourth terms on the right hand side of (4) is zero:

$$\sum_{\alpha=1}^{n} \left(r_\alpha \mathbf{u}_\alpha - r_\alpha \mathbf{v}_\alpha \right) . \mathbf{v}_n = \sum_{\alpha=1}^{n} \left[r_\alpha \mathbf{u}_\alpha - r_\alpha \left(\mathbf{u}_\alpha + \mathbf{v}_n \right) \right] . \mathbf{v}_n$$
$$= - \sum_{\alpha=1}^{n} r_\alpha \mathbf{v}_n^2 = 0 \tag{5}$$

(the last equality follows using (4.20), again).

Combination of (4) and (5) while taking into account that $\mathbf{u}_n = \mathbf{o}$ gives:

$$\sum_{\alpha=1}^{n} \mathbf{k}_\alpha . \mathbf{v}_\alpha + \frac{1}{2} \sum_{\alpha=1}^{n} r_\alpha \mathbf{v}_\alpha^2 = \sum_{\beta=1}^{n-1} \mathbf{k}_\beta . \mathbf{u}_\beta + \frac{1}{2} \sum_{\beta=1}^{n-1} r_\beta \mathbf{u}_\beta^2 . \tag{6}$$

Substitution of (3) and (6), together with the definition (4.19), gives (4.82) immediately.

Verification of the equivalence of traces:

$$\operatorname{tr} \mathbf{D}_\alpha \mathbf{T}_\alpha = (D^{11}T^{11} + D^{12}T^{2} + D^{13}T^{3} + D^{21}T^{2} + D^{22}T^{22} + D^{23}T^{4} + D^{31}T^{3} + D^{32}T^{4} + D^{33}T^{33})_\alpha ,$$

$$\operatorname{tr} \mathbf{T}_\alpha \mathbf{D}_\alpha = (D^{11}T^{11} + D^{21}T^{2} + D^{31}T^{3} + D^{12}T^{2} + D^{22}T^{22} + D^{32}T^{4} + D^{13}T^{3} + D^{23}T^{4} + D^{33}T^{33})_\alpha$$

where $T^{12} = T^{21} \equiv T^2$, $T^{13} = T^{31} \equiv T^3$, $T^{23} = T^{32} \equiv T^4$. The two traces are evidently equal.