Page 176, equation (4.139)

For the sake of future reference the individual terms in (4.139) are numbered as (approximately) indicated below:

$$
\begin{aligned}
T\sigma = &-\Bigg[\sum_{\beta=1}^{n-1}(\overbrace{\sum_{\alpha=1}^{n}\frac{\partial\rho_\alpha\hat{f}_\alpha^{(0)}}{\partial\rho_\beta}}^{1} - \overbrace{\sum_{\alpha=1}^{n}\frac{\partial\rho_\alpha\hat{f}_\alpha^{(0)}}{\partial\rho_n})r_\beta^{(0)}}^{2}\Bigg] - \Bigg\{\overbrace{\sum_{\alpha=1}^{n}\frac{\partial\rho_\alpha\hat{f}_\alpha^{(0)}}{\partial T}}^{3} + \overbrace{\sum_{\alpha=1}^{n}\rho_\alpha s_\alpha^{(0)}}^{4}\Bigg\}\frac{\partial T}{\partial t} \\
&+ \sum_{\gamma=1}^{n}\Bigg[\overbrace{\rho_\gamma\sum_{\alpha=1}^{n}\frac{\partial\rho_\alpha\hat{f}_\alpha^{(0)}}{\partial\rho_\gamma}}^{5} - \overbrace{\rho_\gamma f_\gamma^{(0)}}^{6} - \overbrace{p_\gamma}^{7} - \sum_{\beta=1}^{n-1}(\overbrace{\sum_{\alpha=1}^{n}\frac{\partial\rho_\alpha\hat{f}_\alpha^{(\gamma)}}{\partial\rho_\beta}}^{8} - \overbrace{\sum_{\alpha=1}^{n}\frac{\partial\rho_\alpha\hat{f}_\alpha^{(\gamma)}}{\partial\rho_n})r_\beta^{(0)}}^{9} \\
&- \sum_{\beta=1}^{n-1}(\overbrace{\sum_{\alpha=1}^{n}\frac{\partial\rho_\alpha\hat{f}_\alpha^{(0)}}{\partial\rho_\beta}}^{10} - \overbrace{\sum_{\alpha=1}^{n}\frac{\partial\rho_\alpha\hat{f}_\alpha^{(0)}}{\partial\rho_n})r_\beta^{(\gamma)}}^{11}\Bigg]\mathrm{tr}\mathbf{D}_\gamma - \overbrace{\sum_{\gamma=1}^{n}\Bigg\{\sum_{\alpha=1}^{n}\rho_\alpha f_\alpha^{(\gamma)}\Bigg\}\frac{\partial\mathrm{tr}\mathbf{D}_\gamma}{\partial t}}^{12} \\
&+ \sum_{\gamma=1}^{n}\sum_{\beta=1}^{n-1}\Bigg\{\overbrace{\sum_{\alpha=1}^{n}\frac{\partial\rho_\alpha\hat{f}_\alpha^{(0)}}{\partial\rho_\gamma}\delta_{\beta\gamma}}^{13} - \overbrace{\frac{\partial\rho_\beta\hat{f}_\beta^{(0)}}{\partial\rho_\gamma}}^{14} - \overbrace{\omega_{\beta\gamma}}^{15}\Bigg\}\mathbf{u}_\beta.\mathbf{h}_\gamma - \overbrace{\sum_{\alpha=1}^{n}\{\chi_\alpha/T\}\mathbf{h}_\alpha.\mathbf{g}}^{16} \\
&- \Bigg\{\overbrace{\sum_{\alpha=1}^{n}\frac{\partial\rho_\alpha\hat{f}_\alpha^{(0)}}{\partial T}}^{17} + \overbrace{\sum_{\alpha=1}^{n}\rho_\alpha s_\alpha^{(0)}}^{18}\Bigg\}\mathbf{v}_n.\mathbf{g} - \sum_{\gamma=1}^{n}\Bigg\{\overbrace{\sum_{\alpha=1}^{n}\frac{\partial\rho_\alpha\hat{f}_\alpha^{(\gamma)}}{\partial T}}^{19} + \overbrace{\sum_{\alpha=1}^{n}\rho_\alpha s_\alpha^{(\gamma)}}^{20}\Bigg\}\frac{\partial T}{\partial t}\mathrm{tr}\mathbf{D}_\gamma \\
&- \overbrace{\sum_{\gamma=1}^{n}\sum_{\beta=1}^{n-1}\Big\{\rho_\beta f_\beta^{(\gamma)}\Big\}\mathbf{u}_\beta.\mathrm{grad\ tr}\mathbf{D}_\gamma}^{21} - \overbrace{\sum_{\gamma=1}^{n}\Bigg\{\sum_{\alpha=1}^{n}\rho_\alpha f_\alpha^{(\gamma)}\Bigg\}\mathbf{v}_n.\mathrm{grad\ tr}\mathbf{D}_\gamma}^{22} + \overbrace{(k/T)\mathbf{g}^2}^{23} \\
&+ \sum_{\beta=1}^{n-1}\sum_{\delta=1}^{n-1}(\overbrace{\nu_{\beta\delta}}^{24} - \overbrace{(1/2)r_\beta^{(0)}\delta_{\beta\delta}}^{25})\mathbf{u}_\delta.\mathbf{u}_\beta + \sum_{\beta=1}^{n-1}\Bigg(\overbrace{\frac{\lambda_\beta}{T}}^{26} + \overbrace{\xi_\beta}^{27} - \overbrace{\frac{\partial\rho_\beta\hat{f}_\beta^{(0)}}{\partial T}}^{28} - \overbrace{\rho_\beta s_\beta^{(0)}}^{29}\Bigg)\mathbf{u}_\beta.\mathbf{g} \\
&+ \sum_{\epsilon=1}^{n}\sum_{\gamma=1}^{n}\Bigg[\overbrace{\rho_\epsilon\sum_{\alpha=1}^{n}\frac{\partial\rho_\alpha\hat{f}_\alpha^{(\gamma)}}{\partial\rho_\epsilon}}^{30} - \sum_{\beta=1}^{n-1}\Bigg(\overbrace{\sum_{\alpha=1}^{n}\frac{\partial\rho_\alpha\hat{f}_\alpha^{(\gamma)}}{\partial\rho_\beta}}^{31} - \overbrace{\sum_{\alpha=1}^{n}\frac{\partial\rho_\alpha\hat{f}_\alpha^{(\gamma)}}{\partial\rho_n}}^{32}\Bigg)r_\beta^{(\epsilon)} - \overbrace{\rho_\epsilon f_\epsilon^{(\gamma)}}^{33}
\end{aligned}
$$

$$+\overbrace{\zeta_{\epsilon\gamma}}^{34}\Big]\mathrm{tr}\mathbf{D}_\epsilon\mathrm{tr}\mathbf{D}_\gamma+\overbrace{\sum_{\alpha=1}^{n}\sum_{\gamma=1}^{n}(2\eta_{\alpha\gamma})\mathrm{tr}(\mathring{\mathbf{D}}_\alpha\mathring{\mathbf{D}}_\gamma)}^{35}+\sum_{\epsilon=1}^{n}\sum_{\gamma=1}^{n}\sum_{\beta=1}^{n-1}\Bigg\{\overbrace{\sum_{\alpha=1}^{n}\frac{\partial\rho_\alpha\hat{f}_\alpha^{(\gamma)}}{\partial\rho_\beta}\delta_{\beta\epsilon}}^{36}$$

$$-\overbrace{\frac{\partial\rho_\beta\hat{f}_\beta^{(\gamma)}}{\partial\rho_\epsilon}}^{37}\Bigg\}\mathrm{tr}\mathbf{D}_\gamma(\mathbf{h}_\epsilon.\mathbf{u}_\beta)-\sum_{\beta=1}^{n-1}\sum_{\gamma=1}^{n}\Bigg\{\overbrace{\frac{\partial\rho_\beta\hat{f}_\beta^{(\gamma)}}{\partial T}}^{38}+\overbrace{\rho_\beta s_\beta^{(\gamma)}}^{39}\Bigg\}\mathrm{tr}\mathbf{D}_\gamma(\mathbf{u}_\beta.\mathbf{g})$$

$$-\overbrace{\sum_{\gamma=1}^{n}\sum_{\beta=1}^{n-1}\{\frac{1}{2}r_\beta^{(\gamma)}\}\mathbf{u}_\beta^2\,\mathrm{tr}\mathbf{D}_\gamma}^{40}-\sum_{\gamma=1}^{n}\Bigg\{\overbrace{\sum_{\alpha=1}^{n}\frac{\partial\rho_\alpha\hat{f}_\alpha^{(\gamma)}}{\partial T}}^{41}+\overbrace{\sum_{\alpha=1}^{n}\rho_\alpha s_\alpha^{(\gamma)}}^{42}\Bigg\}\mathrm{tr}\mathbf{D}_\gamma(\mathbf{v}_n.\mathbf{g})\geq 0$$

First, the constitutive equations (4.130)-(4.133) and (4.136)-(4.138) are inserted into (4.89) (remember that grad$T=\mathbf{g}$):

$$\begin{aligned}-T\sigma=&\overbrace{\sum_{\alpha=1}^{n}\frac{\partial}{\partial t}(\rho_\alpha f_\alpha^{(0)}+\sum_{\gamma=1}^{n}\rho_\alpha f_\alpha^{(\gamma)}\mathrm{tr}\mathbf{D}_\gamma)}^{1.1}+\sum_{\alpha=1}^{n}\rho_\alpha(f_\alpha^{(0)}+\sum_{\gamma=1}^{n}f_\alpha^{(\gamma)}\mathrm{tr}\mathbf{D}_\gamma)\mathrm{tr}\mathbf{D}_\alpha\\
&+\overbrace{\sum_{\beta=1}^{n-1}\mathbf{u}_\beta.\mathrm{grad}\big[\rho_\beta(f_\beta^{(0)}+\sum_{\gamma=1}^{n}f_\beta^{(\gamma)}\mathrm{tr}\mathbf{D}_\gamma)\big]}^{8.1}+\overbrace{\mathbf{v}_n.\mathrm{grad}\sum_{\alpha=1}^{n}\rho_\alpha(f_\alpha^{(0)}+\sum_{\gamma=1}^{n}f_\alpha^{(\gamma)}\mathrm{tr}\mathbf{D}_\gamma)}^{9.1}\\
&+\sum_{\alpha=1}^{n}\rho_\alpha(s_\alpha^{(0)}+\sum_{\gamma=1}^{n}s_\alpha^{(\gamma)}\mathrm{tr}\mathbf{D}_\gamma)\frac{\partial T}{\partial t}+\sum_{\beta=1}^{n-1}\rho_\beta(s_\beta^{(0)}+\sum_{\gamma=1}^{n}s_\beta^{(\gamma)}\mathrm{tr}\mathbf{D}_\gamma)\mathbf{u}_\beta.\mathbf{g}\\
&+\mathbf{v}_n.\mathbf{g}\sum_{\alpha=1}^{n}\rho_\alpha(s_\alpha^{(0)}+\sum_{\gamma=1}^{n}s_\alpha^{(\gamma)}\mathrm{tr}\mathbf{D}_\gamma)+(1/T)(-k\mathbf{g}-\sum_{\delta=1}^{n-1}\lambda_\delta\mathbf{u}_\delta+\sum_{\gamma=1}^{n}\chi_\gamma\mathbf{h}_\gamma).\mathbf{g}\\
&-\sum_{\alpha=1}^{n}\mathrm{tr}\big[(\overbrace{-p_\alpha\mathbf{1}}^{4.1}+\overbrace{\sum_{\gamma=1}^{n}\zeta_{\alpha\gamma}(\mathrm{tr}\,\mathbf{D}_\gamma)\mathbf{1}}^{5.1}+\overbrace{\sum_{\gamma=1}^{n}2\eta_{\alpha\gamma}\,\mathring{\mathbf{D}}_\gamma}^{6.1})\,\mathring{\mathbf{D}}_\alpha\,\big]\\
&-(1/3)\sum_{\alpha=1}^{n}\mathrm{tr}(\overbrace{-p_\alpha\mathbf{1}}^{2.1}+\sum_{\gamma=1}^{n}\zeta_{\alpha\gamma}(\mathrm{tr}\,\mathbf{D}_\gamma)\mathbf{1}+\overbrace{\sum_{\gamma=1}^{n}2\eta_{\alpha\gamma}\,\mathring{\mathbf{D}}_\gamma}^{7.1})\mathrm{tr}\,\mathbf{D}_\alpha\\
&+\sum_{\beta=1}^{n-1}(-\xi_\beta\mathbf{g}-\sum_{\delta=1}^{n-1}\nu_{\beta\delta}\mathbf{u}_\delta+\sum_{\gamma=1}^{n}\omega_{\beta\gamma}\mathbf{h}_\gamma).\mathbf{u}_\beta+\frac{1}{2}\sum_{\beta=1}^{n-1}(\overbrace{r_\beta^{(0)}}^{3.1}+\sum_{\gamma=1}^{n}r_\beta^{(\gamma)}\mathrm{tr}\mathbf{D}_\gamma)\mathbf{u}_\beta^2.\end{aligned}\tag{1}$$

A lot of terms from (4.139) can be already seen in (1) in the following order of appearance (remember the reversed signs of $T\sigma$ in (4.139) and here, cf. page 1 above): 6, 33, 4, 20, 29, 39, 18, 42, 23, 26, 16, 34, 27, 24, 15, and 40. The numbered terms in (1) can be further modified as follows.

The time derivatives of responses can be expressed using chain rule. Thus term $\widehat{1.1}$ can be expanded

$$\widehat{1.1} = \sum_{\alpha=1}^{n} \frac{\partial \rho_\alpha f_\alpha^{(0)}}{\partial t} + \sum_{\alpha=1}^{n}\sum_{\gamma=1}^{n} \frac{\partial \rho_\alpha f_\alpha^{(\gamma)}}{\partial t} \mathrm{tr}\mathbf{D}_\gamma + \sum_{\alpha=1}^{n}\sum_{\gamma=1}^{n} \rho_\alpha f_\alpha^{(\gamma)} \frac{\partial\, \mathrm{tr}\mathbf{D}_\gamma}{\partial t}$$

and modified:

$$\widehat{1.1} = \sum_{\alpha=1}^{n} \frac{\partial \rho_\alpha \hat{f}_\alpha^{(0)}}{\partial T}\frac{\partial T}{\partial t} + \sum_{\alpha=1}^{n}\sum_{\gamma=1}^{n} \frac{\partial \rho_\alpha \hat{f}_\alpha^{(0)}}{\partial \rho_\gamma}\frac{\partial \rho_\gamma}{\partial t}$$
$$+ \sum_{\alpha=1}^{n}\sum_{\gamma=1}^{n} \left[\frac{\partial \rho_\alpha \hat{f}_\alpha^{(\gamma)}}{\partial T}\frac{\partial T}{\partial t} + \sum_{\epsilon=1}^{n} \frac{\partial \rho_\alpha \hat{f}_\alpha^{(\gamma)}}{\partial \rho_\epsilon}\frac{\partial \rho_\epsilon}{\partial t} \right] \mathrm{tr}\mathbf{D}_\gamma + \sum_{\alpha=1}^{n}\sum_{\gamma=1}^{n} \rho_\alpha f_\alpha^{(\gamma)} \frac{\partial\, \mathrm{tr}\mathbf{D}_\gamma}{\partial t}. \quad (2)$$

Time derivatives of densities can be expressed from (4.17):

$$\frac{\partial \rho_\alpha}{\partial t} = r_\alpha - \mathrm{div}\rho_\alpha \mathbf{v}_\alpha = r_\alpha - \rho_\alpha \mathrm{div}\mathbf{v}_\alpha - \mathbf{v}_\alpha.\mathrm{grad}\rho_\alpha =$$
$$= r_\alpha - \rho_\alpha \mathrm{tr}\mathbf{D}_\alpha - \mathbf{v}_\alpha.\mathbf{h}_\alpha \qquad \alpha = 1, \ldots, n \quad (3)$$

where (4.123) and (4.8) were used in the last equality. Substitution from (3) into (2) gives:

$$\widehat{1.1} = \sum_{\alpha=1}^{n} \frac{\partial \rho_\alpha \hat{f}_\alpha^{(0)}}{\partial T}\frac{\partial T}{\partial t} + \overbrace{\sum_{\alpha=1}^{n}\sum_{\gamma=1}^{n} \frac{\partial \rho_\alpha \hat{f}_\alpha^{(0)}}{\partial \rho_\gamma}(r_\gamma - \rho_\gamma \mathrm{tr}\mathbf{D}_\gamma - \mathbf{v}_\gamma.\mathbf{h}_\gamma)}^{1a.1}$$
$$+ \sum_{\alpha=1}^{n}\sum_{\gamma=1}^{n} \frac{\partial \rho_\alpha \hat{f}_\alpha^{(\gamma)}}{\partial T}\frac{\partial T}{\partial t}\mathrm{tr}\mathbf{D}_\gamma + \overbrace{\sum_{\alpha=1}^{n}\sum_{\gamma=1}^{n}\sum_{\epsilon=1}^{n} \frac{\partial \rho_\alpha \hat{f}_\alpha^{(\gamma)}}{\partial \rho_\epsilon}(r_\epsilon - \rho_\epsilon \mathrm{tr}\mathbf{D}_\epsilon - \mathbf{v}_\epsilon.\mathbf{h}_\epsilon)\mathrm{tr}\mathbf{D}_\gamma}^{1b.1}$$
$$+ \sum_{\alpha=1}^{n}\sum_{\gamma=1}^{n} \rho_\alpha f_\alpha^{(\gamma)} \frac{\partial\, \mathrm{tr}\mathbf{D}_\gamma}{\partial t}. \quad (4)$$

The non-numbered terms in (4) correspond to following terms in (4.139): 3, 19, 12, resp.

Reaction rates $r_\beta, \beta = 1, \ldots, n-1$ can be substituted from (4.130); the last rate can be expressed from (4.96) using (4.130):

$$r_n = -\sum_{\beta=1}^{n-1} r_\beta = -\sum_{\beta=1}^{n-1} \big(r_\beta^{(0)} + \sum_{\gamma=1}^{n} r_\beta^{(\gamma)} \mathrm{tr}\mathbf{D}_\gamma\big). \tag{5}$$

Then we can write for the terms in (4):

$$\begin{aligned}\widehat{1a.1} &= \sum_{\alpha=1}^{n} \Big(\sum_{\gamma=1}^{n} \frac{\partial \rho_\alpha \hat{f}_\alpha^{(0)}}{\partial \rho_\gamma} r_\gamma - \sum_{\gamma=1}^{n} \frac{\partial \rho_\alpha \hat{f}_\alpha^{(0)}}{\partial \rho_\gamma} \rho_\gamma \mathrm{tr}\mathbf{D}_\gamma - \sum_{\gamma=1}^{n} \frac{\partial \rho_\alpha \hat{f}_\alpha^{(0)}}{\partial \rho_\gamma} \mathbf{v}_\gamma.\mathbf{h}_\gamma \Big) \\ &= \sum_{\alpha=1}^{n} \Big[\sum_{\beta=1}^{n-1} \frac{\partial \rho_\alpha \hat{f}_\alpha^{(0)}}{\partial \rho_\beta} (r_\beta^{(0)} + \sum_{\gamma=1}^{n} r_\beta^{(\gamma)} \mathrm{tr}\mathbf{D}_\gamma) - \frac{\partial \rho_\alpha \hat{f}_\alpha^{(0)}}{\partial \rho_n} \sum_{\beta=1}^{n-1} (r_\beta^{(0)} \\ &+ \sum_{\gamma=1}^{n} r_\beta^{(\gamma)} \mathrm{tr}\mathbf{D}_\gamma) - \sum_{\gamma=1}^{n} \frac{\partial \rho_\alpha \hat{f}_\alpha^{(0)}}{\partial \rho_\gamma} \rho_\gamma \mathrm{tr}\mathbf{D}_\gamma - \overbrace{\sum_{\gamma=1}^{n} \frac{\partial \rho_\alpha \hat{f}_\alpha^{(0)}}{\partial \rho_\gamma} \mathbf{v}_\gamma.\mathbf{h}_\gamma}^{1aa.1} \Big]\end{aligned} \tag{6}$$

and

$$\begin{aligned}\widehat{1b.1} &= \sum_{\alpha=1}^{n} \sum_{\gamma=1}^{n} \Big(\sum_{\epsilon=1}^{n} \frac{\partial \rho_\alpha \hat{f}_\alpha^{(\gamma)}}{\partial \rho_\epsilon} r_\epsilon - \sum_{\epsilon=1}^{n} \frac{\partial \rho_\alpha \hat{f}_\alpha^{(\gamma)}}{\partial \rho_\epsilon} \rho_\epsilon \mathrm{tr}\mathbf{D}_\epsilon - \sum_{\epsilon=1}^{n} \frac{\partial \rho_\alpha \hat{f}_\alpha^{(\gamma)}}{\partial \rho_\epsilon} \mathbf{v}_\epsilon.\mathbf{h}_\epsilon \Big) \mathrm{tr}\mathbf{D}_\gamma \\ &= \sum_{\alpha=1}^{n} \sum_{\gamma=1}^{n} \Big[\sum_{\beta=1}^{n-1} \frac{\partial \rho_\alpha \hat{f}_\alpha^{(\gamma)}}{\partial \rho_\beta} (r_\beta^{(0)} + \sum_{\epsilon=1}^{n} r_\beta^{(\epsilon)} \mathrm{tr}\mathbf{D}_\epsilon) - \frac{\partial \rho_\alpha \hat{f}_\alpha^{(\gamma)}}{\partial \rho_n} (r_\beta^{(0)} \\ &+ \sum_{\epsilon=1}^{n} r_\beta^{(\epsilon)} \mathrm{tr}\mathbf{D}_\epsilon) - \sum_{\epsilon=1}^{n} \frac{\partial \rho_\alpha \hat{f}_\alpha^{(\gamma)}}{\partial \rho_\epsilon} \rho_\epsilon \mathrm{tr}\mathbf{D}_\epsilon - \overbrace{\sum_{\epsilon=1}^{n} \frac{\partial \rho_\alpha \hat{f}_\alpha^{(\gamma)}}{\partial \rho_\epsilon} \mathbf{v}_\epsilon.\mathbf{h}_\epsilon}^{1ba.1} \Big] \mathrm{tr}\mathbf{D}_\gamma\end{aligned} \tag{7}$$

In the second equality of (6) we can identify following terms of (4.139): 1, 10, 2, 11, and 5; in the second equality of (7): 8, 31, 9, 32, and 30.

Several other numbered terms in (1) can be easily modified:

$$\widehat{2.1} = (-1/3) \sum_{\alpha=1}^{n} \mathrm{tr}(-p_\alpha \mathbf{1}) \mathrm{tr}\mathbf{D}_\alpha = (1/3) \mathrm{tr}\mathbf{1} \sum_{\alpha=1}^{n} p_\alpha \mathrm{tr}\mathbf{D}_\alpha = \sum_{\alpha=1}^{n} p_\alpha \mathrm{tr}\mathbf{D}_\alpha \tag{8}$$

which gives 7 in (4.139);

$$\widehat{3.1} = (1/2)\sum_{\beta=1}^{n-1} r_\beta^{(0)}\mathbf{u}_\beta.\mathbf{u}_\beta = (1/2)(r_1^{(0)}\mathbf{u}_1.\mathbf{u}_1 + r_2^{(0)}\mathbf{u}_2.\mathbf{u}_2 + \dots$$

$$+ r_{n-1}^{(0)}\mathbf{u}_{n-1}.\mathbf{u}_{n-1}) \equiv (1/2)\sum_{\delta=1}^{n-1}(r_1^{(0)}\delta_{1\delta}\mathbf{u}_\delta.\mathbf{u}_1) + (1/2)\sum_{\delta=1}^{n-1}(r_2^{(0)}\delta_{2\delta}\mathbf{u}_\delta.\mathbf{u}_2)$$

$$+ \dots (1/2)\sum_{\delta=1}^{n-1}(r_{n-1}^{(0)}\delta_{(n-1)\delta}\mathbf{u}_\delta.\mathbf{u}_{n-1}) = \sum_{\beta=1}^{n-1}\sum_{\delta=1}^{n-1}(1/2)r_\beta^{(0)}\delta_{\beta\delta}\mathbf{u}_\delta.\mathbf{u}_\beta \tag{9}$$

giving 25 in (4.139);

$$\widehat{4.1} = \sum_{\alpha=1}^{n} p_\alpha \mathrm{tr}(\overset{\circ}{\mathbf{D}}_\alpha\ \mathbf{1}) = \sum_{\alpha=1}^{n} p_\alpha \mathrm{tr}\ \overset{\circ}{\mathbf{D}}_\alpha = 0 \tag{10}$$

(remember that $\mathrm{tr}\alpha\mathbf{A} = \alpha\mathrm{tr}\mathbf{A}$ for scalar α) where (4.88) was used in the last equality;

$$\widehat{5.1} = -\sum_{\alpha=1}^{n}\sum_{\gamma=1}^{n}\zeta_{\alpha\gamma}(\mathrm{tr}\mathbf{D}_\gamma)\mathrm{tr}(\overset{\circ}{\mathbf{D}}_\alpha\ \mathbf{1}) = -\sum_{\alpha=1}^{n}\sum_{\gamma=1}^{n}\zeta_{\alpha\gamma}(\mathrm{tr}\mathbf{D}_\gamma)\mathrm{tr}\ \overset{\circ}{\mathbf{D}}_\alpha = 0 \tag{11}$$

where (4.88) was used in the last equality, again;

$$\widehat{6.1} = -\sum_{\alpha=1}^{n}\sum_{\gamma=1}^{n} 2\eta_{\alpha\gamma}\mathrm{tr}(\overset{\circ}{\mathbf{D}}_\gamma\overset{\circ}{\mathbf{D}}_\alpha) \tag{12}$$

which results in term 35 in (4.139); and

$$\widehat{7.1} = -(1/3)\sum_{\alpha=1}^{n}\sum_{\gamma=1}^{n} 2\eta_{\alpha\gamma}\mathrm{tr}\ \overset{\circ}{\mathbf{D}}_\gamma\ \mathrm{tr}\mathbf{D}_\alpha = 0 \tag{13}$$

where (4.88) was used in the last equality, once more.

Before going further, note that gradient of a function $\hat{\varphi}(T, \rho_\gamma)$ can be written in expanded form:

$$\mathrm{grad}\hat{\varphi}(T, \rho_\gamma) = \frac{\partial\hat{\varphi}}{\partial T}\mathrm{grad}T + \sum_{\gamma=1}^{n}\frac{\partial\hat{\varphi}}{\partial\rho_\gamma}\mathrm{grad}\rho_\gamma \equiv \frac{\partial\hat{\varphi}}{\partial T}\mathbf{g} + \sum_{\gamma=1}^{n}\frac{\partial\hat{\varphi}}{\partial\rho_\gamma}\mathbf{h}_\gamma.$$

Gradients appearing in term 8.1 of (1) can be thus written:

$$
\begin{aligned}
\mathrm{grad}\rho_\beta f_\beta^{(0)} + \sum_{\gamma=1}^{n} \mathrm{grad}(\rho_\beta f_\beta^{(\gamma)} \mathrm{tr}\mathbf{D}_\gamma) &= \rho_\beta \mathrm{grad} f_\beta^{(0)} + f_\beta^{(0)} \mathbf{h}_\beta \\
&+ \sum_{\gamma=1}^{n} (\rho_\beta f_\beta^{(\gamma)} \mathrm{grad}\mathrm{tr}\mathbf{D}_\gamma + \mathrm{tr}\mathbf{D}_\gamma \mathrm{grad}\rho_\beta f_\beta^{(\gamma)}) \\
&= \frac{\partial \rho_\beta \hat{f}_\beta^{(0)}}{\partial T}\mathbf{g} + \sum_{\gamma=1}^{n} \frac{\partial \rho_\beta \hat{f}_\beta^{(0)}}{\partial \rho_\gamma}\mathbf{h}_\gamma \\
&+ \sum_{\gamma=1}^{n} \rho_\beta f_\beta^{(\gamma)} \mathrm{grad}\,\mathrm{tr}\mathbf{D}_\gamma + \sum_{\gamma=1}^{n} \mathrm{tr}\mathbf{D}_\gamma \mathrm{grad}\rho_\beta f_\beta^{(\gamma)} \qquad (14)
\end{aligned}
$$

and then 8.1 is as follows:

$$
\begin{aligned}
\widehat{8.1} = \sum_{\beta=1}^{n-1} \Big(& \frac{\partial \rho_\beta \hat{f}_\beta^{(0)}}{\partial T} \mathbf{u}_\beta . \mathbf{g} + \sum_{\gamma=1}^{n} \frac{\partial \rho_\beta \hat{f}_\beta^{(0)}}{\partial \rho_\gamma} \mathbf{u}_\beta . \mathbf{h}_\gamma + \sum_{\gamma=1}^{n} \rho_\beta f_\beta^{(\gamma)} \mathbf{u}_\beta . \mathrm{grad}\,\mathrm{tr}\mathbf{D}_\gamma \\
& + \overbrace{\sum_{\gamma=1}^{n} \mathrm{tr}\mathbf{D}_\gamma\, \mathbf{u}_\beta . \mathrm{grad}\rho_\beta f_\beta^{(\gamma)}}^{8a.1} \Big). \qquad (15)
\end{aligned}
$$

Terms 28, 14, and 21 from (4.139) can be seen in (15). Term $8a.1$ in (15) can be further modified:

$$
\begin{aligned}
\widehat{8a.1} &= \sum_{\beta=1}^{n-1} \sum_{\gamma=1}^{n} \mathrm{tr}\mathbf{D}_\gamma\, \mathbf{u}_\beta . \Big(\frac{\partial \rho_\beta \hat{f}_\beta^{(\gamma)}}{\partial T} \mathbf{g} + \sum_{\alpha=1}^{n} \frac{\partial \rho_\beta \hat{f}_\beta^{(\gamma)}}{\partial \rho_\alpha} . \mathbf{h}_\alpha \Big) \\
&= \sum_{\beta=1}^{n-1} \sum_{\gamma=1}^{n} \mathrm{tr}\mathbf{D}_\gamma \Big(\frac{\partial \rho_\beta \hat{f}_\beta^{(\gamma)}}{\partial T} \mathbf{u}_\beta . \mathbf{g} + \sum_{\alpha=1}^{n} \frac{\partial \rho_\beta \hat{f}_\beta^{(\gamma)}}{\partial \rho_\alpha} \mathbf{u}_\beta . \mathbf{h}_\alpha \Big). \qquad (16)
\end{aligned}
$$

The last right hand side of (16) contains terms 38 and 37 from (4.139).

Finally, term 9.1 in (1) will be modified by the expansion of gradients:

$$
\begin{aligned}
\widehat{9.1} &= \mathbf{v}_n . \sum_{\alpha=1}^{n} \mathrm{grad}\rho_\alpha f_\alpha^{(0)} + \mathbf{v}_n . \sum_{\alpha=1}^{n} \sum_{\gamma=1}^{n} \mathrm{grad}(\rho_\alpha f_\alpha^{(\gamma)} \mathrm{tr}\mathbf{D}_\gamma) \\
&= \mathbf{v}_n . \Big(\sum_{\alpha=1}^{n} \frac{\partial \rho_\alpha \hat{f}_\alpha^{(0)}}{\partial T} \mathbf{g} + \sum_{\alpha=1}^{n} \sum_{\gamma=1}^{n} \frac{\partial \rho_\alpha \hat{f}_\alpha^{(0)}}{\partial \rho_\alpha} \mathbf{h}_\gamma \Big)
\end{aligned}
$$

$$+\mathbf{v}_n.\sum_{\alpha=1}^{n}\sum_{\gamma=1}^{n}\rho_\alpha f_\alpha^{(\gamma)}\mathrm{grad}\,\mathrm{tr}\mathbf{D}_\gamma+\mathbf{v}_n.\sum_{\alpha=1}^{n}\sum_{\gamma=1}^{n}\mathrm{tr}\mathbf{D}_\gamma\mathrm{grad}\rho_\alpha f_\alpha^{(\gamma)}$$

$$=\sum_{\alpha=1}^{n}\frac{\partial\rho_\alpha\hat{f}_\alpha^{(0)}}{\partial T}\mathbf{v}_n.\mathbf{g}+\sum_{\alpha=1}^{n}\sum_{\gamma=1}^{n}\frac{\partial\rho_\alpha\hat{f}_\alpha^{(0)}}{\partial\rho_\gamma}\mathbf{v}_n.\mathbf{h}_\gamma+\sum_{\alpha=1}^{n}\sum_{\gamma=1}^{n}\rho_\alpha f_\alpha^{(\gamma)}\mathbf{v}_n.\mathrm{grad}\,\mathrm{tr}\mathbf{D}_\gamma$$

$$+\mathbf{v}_n.\sum_{\alpha=1}^{n}\sum_{\gamma=1}^{n}\mathrm{tr}\mathbf{D}_\gamma\Big(\frac{\partial\rho_\alpha\hat{f}_\alpha^{(\gamma)}}{\partial T}\mathbf{g}+\sum_{\epsilon=1}^{n}\frac{\partial\rho_\alpha\hat{f}_\alpha^{(\gamma)}}{\partial\rho_\epsilon}\mathbf{h}_\epsilon\Big)=\sum_{\alpha=1}^{n}\frac{\partial\rho_\alpha\hat{f}_\alpha^{(0)}}{\partial T}\mathbf{v}_n.\mathbf{g}$$

$$+\overbrace{\sum_{\alpha=1}^{n}\sum_{\gamma=1}^{n}\frac{\partial\rho_\alpha\hat{f}_\alpha^{(0)}}{\partial\rho_\gamma}\mathbf{v}_n.\mathbf{h}_\gamma}^{9a.1}+\sum_{\alpha=1}^{n}\sum_{\gamma=1}^{n}\rho_\alpha f_\alpha^{(\gamma)}\mathbf{v}_n.\mathrm{grad}\,\mathrm{tr}\mathbf{D}_\gamma$$

$$+\sum_{\alpha=1}^{n}\sum_{\gamma=1}^{n}\mathrm{tr}\mathbf{D}_\gamma\frac{\partial\rho_\alpha\hat{f}_\alpha^{(\gamma)}}{\partial T}\mathbf{v}_n.\mathbf{g}+\overbrace{\sum_{\alpha=1}^{n}\sum_{\gamma=1}^{n}\sum_{\epsilon=1}^{n}\mathrm{tr}\mathbf{D}_\gamma\frac{\partial\rho_\alpha\hat{f}_\alpha^{(\gamma)}}{\partial\rho_\epsilon}\mathbf{v}_n.\mathbf{h}_\epsilon}^{9b.1}. \qquad (17)$$

In the last equality of (17) following terms from (4.139) appear: 17, 22, and 41.

Combining $1aa.1$ from (6) with $9a.1$ in (17) we obtain:

$$\overbrace{1aa.1}+\overbrace{9a.1}=\sum_{\alpha=1}^{n}\sum_{\gamma=1}^{n}\frac{\partial\rho_\alpha\hat{f}_\alpha^{(0)}}{\partial\rho_\gamma}(\mathbf{v}_n-\mathbf{v}_\gamma).\mathbf{h}_\gamma=-\sum_{\alpha=1}^{n}\sum_{\gamma=1}^{n}\frac{\partial\rho_\alpha\hat{f}_\alpha^{(0)}}{\partial\rho_\gamma}\mathbf{u}_\gamma.\mathbf{h}_\gamma. \quad (18)$$

The last term in (18) should now be expressed with the aid of the product $\mathbf{u}_\beta.\mathbf{h}_\gamma$. Subscripts at members of that term are always equal ($\mathbf{u}_\gamma.\mathbf{h}_\gamma$) therefore this can be easily arranged using Kronecker's δ (more detailed check is at the end of this document):

$$-\sum_{\alpha=1}^{n}\sum_{\gamma=1}^{n}\frac{\partial\rho_\alpha\hat{f}_\alpha^{(0)}}{\partial\rho_\gamma}\mathbf{u}_\gamma.\mathbf{h}_\gamma\equiv-\sum_{\gamma=1}^{n}\sum_{\beta=1}^{n-1}\sum_{\alpha=1}^{n}\frac{\partial\rho_\alpha\hat{f}_\alpha^{(0)}}{\partial\rho_\gamma}\delta_{\beta\gamma}\mathbf{u}_\beta.\mathbf{h}_\gamma. \qquad (19)$$

The right hand side of (19) gives term 13 in (4.139). Similarly, $1ba.1$ from (7) can be combined with $9b.1$ in (17) and modified giving:

$$\overbrace{1ba.1}+\overbrace{9b.1}=\sum_{\alpha=1}^{n}\sum_{\gamma=1}^{n}\sum_{\epsilon=1}^{n}\mathrm{tr}\mathbf{D}_\gamma\frac{\partial\rho_\alpha\hat{f}_\alpha^{(\gamma)}}{\partial\rho_\epsilon}(\mathbf{v}_n-\mathbf{v}_\epsilon).\mathbf{h}_\epsilon$$

$$=-\sum_{\alpha=1}^{n}\sum_{\gamma=1}^{n}\sum_{\epsilon=1}^{n}\mathrm{tr}\mathbf{D}_\gamma\frac{\partial\rho_\alpha\hat{f}_\alpha^{(\gamma)}}{\partial\rho_\epsilon}\mathbf{u}_\epsilon.\mathbf{h}_\epsilon$$

$$=-\sum_{\epsilon=1}^{n}\sum_{\gamma=1}^{n}\sum_{\beta=1}^{n-1}\sum_{\alpha=1}^{n}\frac{\partial\rho_\alpha\hat{f}_\alpha^{(\gamma)}}{\partial\rho_\epsilon}\delta_{\beta\epsilon}\mathrm{tr}\mathbf{D}_\gamma\mathbf{u}_\beta.\mathbf{h}_\epsilon. \qquad (20)$$

The last term in (20) leads to term 36 in (4.139) (due to the presence of $\delta_{\beta\epsilon}$ it makes no difference if the partial differentiation is made with respect to ρ_β or ρ_ϵ).

Thus all 42 terms in (4.139) are recovered.

Appendix. Check for the correctness of (19).

Remember that $\mathbf{u}_n = \mathbf{o}$, cf. (4.24).

$$
\begin{aligned}
&-\sum_{\alpha=1}^{n}\sum_{\gamma=1}^{n}\frac{\partial\rho_\alpha\hat{f}_\alpha^{(0)}}{\partial\rho_\gamma}\mathbf{u}_\gamma.\mathbf{h}_\gamma\\
&=-\sum_{\gamma=1}^{n}\frac{\partial\rho_1\hat{f}_1^{(0)}}{\partial\rho_\gamma}\mathbf{u}_\gamma.\mathbf{h}_\gamma-\sum_{\gamma=1}^{n}\frac{\partial\rho_2\hat{f}_2^{(0)}}{\partial\rho_\gamma}\mathbf{u}_\gamma.\mathbf{h}_\gamma-\ldots-\sum_{\gamma=1}^{n}\frac{\partial\rho_n\hat{f}_n^{(0)}}{\partial\rho_\gamma}\mathbf{u}_\gamma.\mathbf{h}_\gamma\\
&=-\Big(\frac{\partial\rho_1\hat{f}_1^{(0)}}{\partial\rho_1}\mathbf{u}_1.\mathbf{h}_1+\frac{\partial\rho_1\hat{f}_1^{(0)}}{\partial\rho_2}\mathbf{u}_2.\mathbf{h}_2+\ldots+\frac{\partial\rho_1\hat{f}_1^{(0)}}{\partial\rho_{n-1}}\mathbf{u}_{n-1}.\mathbf{h}_{n-1}\Big)\\
&-\Big(\frac{\partial\rho_2\hat{f}_2^{(0)}}{\partial\rho_1}\mathbf{u}_1.\mathbf{h}_1+\frac{\partial\rho_2\hat{f}_2^{(0)}}{\partial\rho_2}\mathbf{u}_2.\mathbf{h}_2+\ldots+\frac{\partial\rho_2\hat{f}_2^{(0)}}{\partial\rho_{n-1}}\mathbf{u}_{n-1}.\mathbf{h}_{n-1}\Big)-\ldots\\
&-\Big(\frac{\partial\rho_n\hat{f}_n^{(0)}}{\partial\rho_1}\mathbf{u}_1.\mathbf{h}_1+\frac{\partial\rho_n\hat{f}_n^{(0)}}{\partial\rho_2}\mathbf{u}_2.\mathbf{h}_2+\ldots+\frac{\partial\rho_n\hat{f}_n^{(0)}}{\partial\rho_{n-1}}\mathbf{u}_{n-1}.\mathbf{h}_{n-1}\Big)
\end{aligned}
\tag{21}
$$

$$
\begin{aligned}
&-\sum_{\gamma=1}^{n}\sum_{\beta=1}^{n-1}\sum_{\alpha=1}^{n}\frac{\partial\rho_\alpha\hat{f}_\alpha^{(0)}}{\partial\rho_\gamma}\delta_{\beta\gamma}\mathbf{u}_\beta.\mathbf{h}_\gamma\\
&=-\sum_{\gamma=1}^{n}\sum_{\alpha=1}^{n}\frac{\partial\rho_\alpha\hat{f}_\alpha^{(0)}}{\partial\rho_\gamma}\delta_{1\gamma}\mathbf{u}_1.\mathbf{h}_\gamma-\sum_{\gamma=1}^{n}\sum_{\alpha=1}^{n}\frac{\partial\rho_\alpha\hat{f}_\alpha^{(0)}}{\partial\rho_\gamma}\delta_{2\gamma}\mathbf{u}_2.\mathbf{h}_\gamma-\ldots\\
&-\sum_{\gamma=1}^{n}\sum_{\alpha=1}^{n}\frac{\partial\rho_\alpha\hat{f}_\alpha^{(0)}}{\partial\rho_\gamma}\delta_{(n-1)\gamma}\mathbf{u}_{n-1}.\mathbf{h}_\gamma\\
&=-\sum_{\alpha=1}^{n}\frac{\partial\rho_\alpha\hat{f}_\alpha^{(0)}}{\partial\rho_1}\mathbf{u}_1.\mathbf{h}_1-\sum_{\alpha=1}^{n}\frac{\partial\rho_\alpha\hat{f}_\alpha^{(0)}}{\partial\rho_2}\mathbf{u}_2.\mathbf{h}_2-\ldots-\sum_{\alpha=1}^{n}\frac{\partial\rho_\alpha\hat{f}_\alpha^{(0)}}{\partial\rho_{n-1}}\mathbf{u}_{n-1}.\mathbf{h}_{n-1}.
\end{aligned}
\tag{22}
$$

Evidently, (21) is equal to (22) as was to be checked.