Page 95, equation (3.98)

To derive this equation we begin with deciphering the product $\mathbf{v}.\mathbf{Tn}$:

$$\begin{aligned}\mathbf{v}.\mathbf{Tn} &= v^1(\mathbf{Tn})^1 + v^2(\mathbf{Tn})^2 + v^3(\mathbf{Tn})^3 \\ &= v^1T^{11}n^1 + v^1T^{12}n^2 + v^1T^{13}n^3 \\ &+ v^2T^{21}n^1 + v^2T^{22}n^2 + v^2T^{23}n^3 \\ &+ v^3T^{31}n^1 + v^3T^{32}n^2 + v^3T^{33}n^3 \\ &= n^1\left(\sum_i v^iT^{i1}\right) + n^2\left(\sum_i v^iT^{i2}\right) + n^3\left(\sum_i v^iT^{i3}\right) \\ &= (\mathbf{vT}).\mathbf{n}\end{aligned}$$

where the components of the vector $\mathbf{vT}$ are $(\mathbf{vT})^j = \sum_i v^iT^{ij}$. Then the first term on the left-hand side of (3.98) can be modified applying Gauss' theorem, or (3.23):

$$\int_{\partial\mathcal{V}} \mathbf{v}.\mathbf{Tn}\,\mathrm{d}a = \int_{\partial\mathcal{V}} (\mathbf{vT}).\mathbf{n}\,\mathrm{d}a = \int_{\mathcal{V}} \mathrm{div}(\mathbf{vT})\,\mathrm{d}v \tag{1}$$

The component form of the divergence in equation (1) is as follows:

$$\begin{aligned}\mathrm{div}(\mathbf{vT}) &= \frac{\partial}{\partial x^1}\left(v^1T^{11} + v^2T^{21} + v^3T^{31}\right) \\ &+ \frac{\partial}{\partial x^2}\left(v^1T^{12} + v^2T^{22} + v^3T^{32}\right) \\ &+ \frac{\partial}{\partial x^3}\left(v^1T^{13} + v^2T^{23} + v^3T^{33}\right) \\ &= T^{11}\frac{\partial v^1}{\partial x^1} + T^{21}\frac{\partial v^2}{\partial x^1} + T^{31}\frac{\partial v^3}{\partial x^1} + v^1\frac{\partial T^{11}}{\partial x^1} + v^2\frac{\partial T^{21}}{\partial x^1} + v^3\frac{\partial T^{31}}{\partial x^1} \\ &= T^{12}\frac{\partial v^1}{\partial x^2} + T^{22}\frac{\partial v^2}{\partial x^2} + T^{32}\frac{\partial v^3}{\partial x^2} + v^1\frac{\partial T^{12}}{\partial x^2} + v^2\frac{\partial T^{22}}{\partial x^2} + v^3\frac{\partial T^{32}}{\partial x^2} \\ &= T^{13}\frac{\partial v^1}{\partial x^3} + T^{23}\frac{\partial v^2}{\partial x^3} + T^{33}\frac{\partial v^3}{\partial x^3} + v^1\frac{\partial T^{13}}{\partial x^3} + v^2\frac{\partial T^{23}}{\partial x^3} + v^3\frac{\partial T^{33}}{\partial x^3}\end{aligned} \tag{2}$$

Remind that:

$$L^{ij} = (\mathrm{grad}\,\mathbf{v})^{ij} = \frac{\partial v^i}{\partial x^j}; \qquad (\mathrm{div}\,\mathbf{T})^j = \sum_k \frac{\partial T^{jk}}{\partial x^k}$$

Taking into account the symmetry of $\mathbf{T}$ – equation (3.93) – it follows from equation (2):

$$\begin{aligned}\operatorname{div}(\mathbf{vT}) &= \sum_i \left(L^{1i}T^{1i}\right) + \sum_i \left(L^{2i}T^{2i}\right) + \sum_i \left(L^{3i}T^{3i}\right) + \sum_j v^j (\operatorname{div}\mathbf{T})^j \\ &= \operatorname{tr}(\mathbf{LT}) + \mathbf{v}.\operatorname{div}\mathbf{T} \end{aligned} \tag{3}$$

Applying equations (1) and (3) in the left-hand side of (3.98) following integral is obtained:

$$\int_{\mathcal{V}} \left[\mathbf{v}.(\operatorname{div}\mathbf{T} + \rho\underline{\beta}) + \operatorname{tr}(\mathbf{LT})\right] \, \mathrm{d}v \tag{4}$$

The first addend in equation (4) is zero – cf. (3.89); the second addend can be modified using the decomposition (3.15) and the fact that the trace of the product of a symmetric ($\mathbf{T}$) and asymmetric ($\mathbf{W}$) tensors is zero. Thus

$$\operatorname{tr}(\mathbf{LT}) = \operatorname{tr}[(\mathbf{D}+\mathbf{W})\mathbf{T}] = \operatorname{tr}(\mathbf{DT}) + \operatorname{tr}(\mathbf{WT}) = \operatorname{tr}(\mathbf{DT}) \tag{5}$$

Introducing (5) into (4), equation (3.98) follows.