Page 97, equation (3.107)

To begin the proof we multiply equation (3.78) by the velocity vector:

$$\rho\dot{\mathbf{v}}.\mathbf{v} = (\mathrm{div}\mathbf{T}).\mathbf{v} + \rho(\mathbf{b}+\mathbf{i}).\mathbf{v}$$

and modify the result into the "balance of kinetic energy":

$$\mathbf{v}.\left[\rho\dot{\mathbf{v}} - \mathrm{div}\mathbf{T} - \rho(\mathbf{b}+\mathbf{i})\right] = 0 \tag{1}$$

The derivative in equation (3.106) can be written

$$\dot{\overline{u + \tfrac{1}{2}\mathbf{v}^2}} = \dot{u} + \mathbf{v}.\dot{\mathbf{v}} \tag{2}$$

as follows directly from (3.8).

The divergence term in (3.106) can be modified using the definition of the velocity gradient $\mathbf{L}$, equation (3.14); the component form of vectors or tensors is usually more instructive (the summation convention is supposed):

$$\mathrm{div}(\mathbf{vT}) \equiv \frac{\partial v^i T^{ij}}{\partial x^j} = T^{ij}\frac{\partial v^i}{\partial x^j} + v^i\frac{\partial T^{ij}}{\partial x^j} = T^{ij}L^{ij} + v^i\frac{\partial T^{ij}}{\partial x^j} = \mathrm{tr}(\mathbf{L}\mathbf{T}^T) + \mathbf{v}.\mathrm{div}\mathbf{T} \tag{3}$$

The trace term can be modified using the decomposition of the velocity gradient $\mathbf{L}$ into the stretching $\mathbf{D}$ and spin $\mathbf{W}$, cf. equation (3.15), taking into account the symmetry of the stress sensor, equation (3.93):

$$\mathrm{tr}(\mathbf{L}\mathbf{T}^T) \equiv L^{ij}T^{ij} = (D^{ij} + W^{ij})T^{ij} = T^{ji}(D^{ij} + W^{ij})$$

It can be easily verified that the trace of product of the symmetric and asymmetric tensor, $\mathbf{T}$ and $\mathbf{W}$ in our case, respectively, is zero. Taking further into account the symmetry of $\mathbf{D}$ (and (3.93) once more) we can write:

$$T^{ji}(D^{ij} + W^{ij}) = T^{ji}D^{ij} = T^{ij}D^{ji} = \mathrm{tr}(\mathbf{TD})$$

Equation (3) can be thus finally written:

$$\mathrm{div}(\mathbf{vT}) = \mathrm{tr}(\mathbf{TD}) + \mathbf{v}.\mathrm{div}\mathbf{T} \tag{4}$$

Introducing (2) and (4) into (3.106) results in:

$$\rho\dot{u} + \rho\mathbf{v}.\dot{\mathbf{v}} = -\mathrm{div}\mathbf{q} + Q + \mathrm{tr}(\mathbf{TD}) + \mathbf{v}.\mathrm{div}\mathbf{T} + \rho(\mathbf{b}+\mathbf{i}).\mathbf{v}$$

and finally:

$$\rho\dot{u} + \mathbf{v}.\left[\rho\dot{\mathbf{v}} - \mathrm{div}\mathbf{T} - \rho(\mathbf{b}+\mathbf{i})\right] = -\mathrm{div}\mathbf{q} + Q + \mathrm{tr}(\mathbf{TD}) \tag{5}$$

Equation (3.107) is obtained combining (5) and (1).