Volume work

Some comments on the volume work which is well known from classical (reversible) thermodynamics; see also footnote 9 on page 10 in the book. The core motivation of this note is the local (internal) energy balance (3.107).

Let us analyze simple case where the stress tensor would be given only by the following simple expression: $\mathbf{T} = -P\mathbf{1}$; then, using also (3.16), we have:

$$\mathrm{tr}(\mathbf{TD}) = -P\,\mathrm{tr}\mathbf{D} = -P\,\mathrm{tr}\mathbf{L} = -P\,\mathrm{div}\mathbf{v}. \tag{1}$$

From the local form of the mass balance (3.63) we have:

$$\mathrm{div}\mathbf{v} = -\dot{\rho}/\rho. \tag{2}$$

Combination of (1) and (2) gives

$$\mathrm{tr}(\mathbf{TD}) = P\dot{\rho}/\rho. \tag{3}$$

From the definition of the specific volume (3.199) we obtain the following relationship between the time derivatives:

$$\dot{\rho} = \dot{(\overline{1/v})} = -(1/v)^2\dot{v} = -\rho^2\dot{v}. \tag{4}$$

Introducing (4) into (3):

$$\mathrm{tr}(\mathbf{TD}) = -P\,\rho\dot{v}. \tag{5}$$

The right hand side of (5) represents the volume work and the balance (3.107) can be rewritten in this case:

$$\rho\dot{u} = -\mathrm{div}\mathbf{q} + Q - P\,\rho\dot{v},$$

$$\dot{u} = v(-\mathrm{div}\mathbf{q} + Q) - P\,\dot{v}. \tag{6}$$

Equation (6) is the first law of classical (equilibrium) thermodynamics (2.1) written in specific quantities (classically, the surface and volume heating are not distinguished). The (classical) volume work thus make sense only in the case of very specific (and restricted) type of the stress tensor, see above. In other words, in the case of the model of the thermoelastic simple fluid, cf. (3.181); in more complex cases the stress is not given just by the pressure, see, e.g., (3.195). Apparently equivalent equilibrium expression as (3.168) cannot be applied, because in equilibrium there is no (specific) volume change - cf. the equilibrium condition $(3.159)_1$ and (5).