Exercise 4 to section 3.1[1]

Show that the quadratic form $\mathbf{W}\mathbf{x} \,.\, \mathbf{x} = \sum_{i,j} W^{ij} x^j x^i$ (in cartesian coordinates), where $\mathbf{W}$ is the skew-symmetric matrix (tensor), is zero. Try to answer before continuing reading.

Skew-symmetric matrix fulfills by definition $W^{ij} + W^{ji} = 0$ which also means that $W^{ii} = 0$. Then

$$\begin{aligned}\mathbf{W}\mathbf{x} \,.\, \mathbf{x} &= \sum_i (W^{i1} x^1 x^i + W^{i2} x^2 x^i + W^{i3} x^3 x^i) \\ &= W^{11}(x^1)^2 + W^{12} x^2 x^1 + W^{13} x^3 x^1 + W^{21} x^1 x^2 + W^{22}(x^2)^2 \\ &+ W^{23} x^3 x^2 + W^{31} x^1 x^3 + W^{32} x^2 x^3 + W^{33}(x^3)^2 \\ &= 0 + W^{12} x^2 x^1 + W^{13} x^3 x^1 - W^{12} x^1 x^2 + 0 + W^{23} x^3 x^2 \\ &- W^{13} x^1 x^3 - W^{23} x^2 x^3 + 0 = 0.\end{aligned}$$

[1] Based on I. Samohýl: Irreversible Thermodynamics. Prague: University of Chemical Technology, 1998 (*in Czech*).