Exercise 2 to section 3.1[1]

Motion $\mathbf{x} = \underline{\chi}(\mathbf{X}, t)$ (p. 68) governed by the equation $\mathbf{x} = \mathbf{A}\mathbf{X}$ where the tensor $\mathbf{A}$ is expressed by the matrix

$$\begin{bmatrix} 1 & 0 & 0 \\ \alpha t & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{1}$$

represents a *simple shear* (α is some constant). Note that the tensor $\mathbf{A}$ is a function of time.

Let us select identical cartesian frames for the reference and actual configurations (schematically $i \equiv J = 1, 2, 3$). What is the component representation of this motion, of the inverse motion and what are the motion-related (kinematic) fields? Try to answer before continuing reading.

The component representation, $x^i = \chi^i(X^i, t)$, of simple shear is:

$$x^1 = X^1,\ x^2 = \alpha t X^1 + X^2,\ x^3 = X^3. \tag{2}$$

From (2) we can easily found the inverse motion (p. 68) in the component form, $X^i = \chi^{-1i}(x^i, t)$:

$$X^1 = x^1,\ X^2 = x^2 - \alpha t X^1,\ X^3 = x^3. \tag{3}$$

Its vectorial representation is $\mathbf{X} = \mathbf{A}^{-1}\mathbf{x}$ where the matrix (tensor) $\mathbf{A}^{-1}$ is:

$$\begin{bmatrix} 1 & 0 & 0 \\ -\alpha t & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}. \tag{4}$$

The velocity (p. 69) is given by $v^i = \partial\chi^i/\partial t$ and from it $v^1 = 0, v^2 = \alpha X^1, v^3 = 0$ or in the vectorial form:

$$\mathbf{v} = \frac{\partial \mathbf{A}}{\partial t}\mathbf{X} = \begin{bmatrix} 0 & 0 & 0 \\ \alpha & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \mathbf{X}. \tag{5}$$

The deformation gradient (p. 69) is given by $F^{ij} = \partial\chi^i/\partial X^j$ and from this we see that $\mathbf{F} = \mathbf{A}$ and also that $\dot{\mathbf{F}}$ is given by the matrix shown in (5). It is also obvious that $\mathbf{F}^{-1} = \mathbf{A}^{-1}$.

[1]Based on I. Samohýl: Irreversible Thermodynamics. Prague: University of Chemical Technology, 1998 (*in Czech*).

From the definition of the velocity gradient (p. 70), $L^{ij} = \partial v^i / \partial x^j$, it follows that it is also given by the matrix shown in (5), which also means that $\mathbf{L} \equiv \dot{\mathbf{F}}$. Then it is easy to verify the identity $\mathbf{L} = \dot{\mathbf{F}}\mathbf{F}^{-1}$.

The decomposition to the stretching ($\mathbf{D}$) and spin ($\mathbf{W}$; p. 70) is:

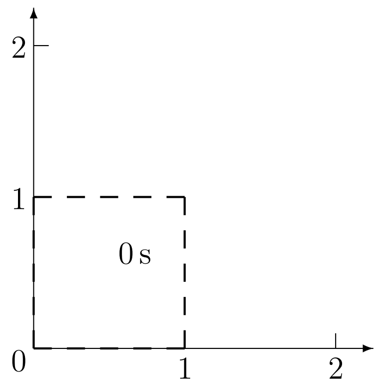
$$\mathbf{D} = \tfrac{1}{2}(\mathbf{L} + \mathbf{L}^T) = \begin{bmatrix} 0 & \alpha/2 & 0 \\ \alpha/2 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \tag{6}$$

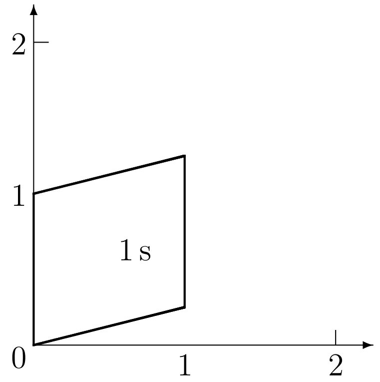
$$\mathbf{W} = \tfrac{1}{2}(\mathbf{L} - \mathbf{L}^T) = \begin{bmatrix} 0 & -\alpha/2 & 0 \\ \alpha/2 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}. \tag{7}$$

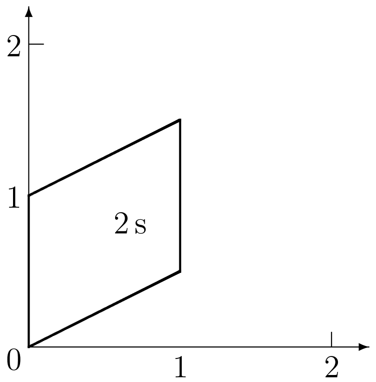
).

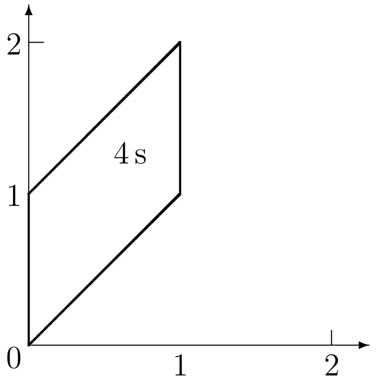
The determinant of the deformation gradient (p. 70) is $J \equiv |\det \mathbf{F}| = 1$. The density thus remains constant and equal to the referential density (p. 87): $\rho = \rho_0 / J = \rho_0$.

Let us outline some results for the specific case of $\alpha = \frac{1}{4}\,\mathrm{s}^{-1}$ and the shape change of a cube, initially residing at the positive corner of the coordinate axes (the referential configuration) during the first 4 s of motion. Only its bottom face is shown:






2 s
0
1
2


4 s
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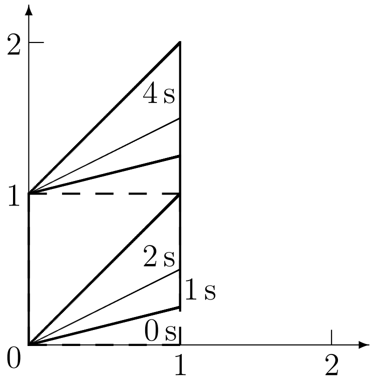
4 s
2 s
1 s
0 s
0
1
2