Exercise 1 to section 3.1[1]

Let us analyze a simple motion $\mathbf{x} = \underline{\chi}(\mathbf{X}, t)$ (p. 68) in the form

$$\mathbf{x} = \mathbf{b} + \alpha t \mathbf{X}; \quad t > 0, \tag{1}$$

where $\mathbf{b}$ and α is a constant vector and (non-zero) scalar, respectively. When $\alpha t > 1$ we call this motion the *volume expansion* whereas the case $\alpha t < 1$ represents the *volume contraction* (the *compression*). Let us further select identical cartesian frames for the reference and actual configurations (schematically $i \equiv J = 1, 2, 3$). What is the component representation of this motion, of the inverse motion and what are the motion-related (kinematic) fields? Try to answer before continuing reading.

The component representation of motion (1) is simply:

$$x^i = b^i + \alpha t X^i = \chi^i(X^i, t). \tag{2}$$

From (2) we can easily found the inverse motion (p. 68) in the component

$$X^i = (x^i - b^i)/\alpha t \equiv \chi^{-1i}(x^i, t) \tag{3}$$

and the vectorial

$$\mathbf{X} = (\mathbf{x} - \mathbf{b})/\alpha t \equiv \underline{\chi}^{-1}(\mathbf{x}, t) \tag{4}$$

forms.

The velocity (p. 69) is given by

$$v^i = \partial \chi^i / \partial t = \alpha X^i \quad \text{or} \quad \mathbf{v} = \alpha \mathbf{X}. \tag{5}$$

The deformation gradient (p. 69) is

$$F^{ij} = \partial \chi^i / \partial X^j = \begin{cases} \alpha t & \text{for} \quad i = j, \\ 0 & \text{for} \quad i \neq j. \end{cases} \tag{6}$$

In the vectorial representation:

$$\mathbf{F} = \alpha t \mathbf{1} \Rightarrow \mathbf{F}^{-1} = (\alpha t)^{-1} \mathbf{1}; \quad \dot{\mathbf{F}} = \alpha \mathbf{1}. \tag{7}$$

The velocity gradient (p. 70):

$$L^{ij} = \partial v^i / \partial x^j = \alpha \partial X^i / \partial x^j = \begin{cases} \alpha(1/\alpha t) = t^{-1} & \text{for} \quad i = j, \\ 0 & \text{for} \quad i \neq j \end{cases} \tag{8}$$

[1] Based on I. Samohýl: Irreversible Thermodynamics. Prague: University of Chemical Technology, 1998 (*in Czech*).

and in the vectorial representation:

$$\mathbf{L} = t^{-1}\mathbf{1} \equiv \dot{\mathbf{F}}\mathbf{F}^{-1} \tag{9}$$

which confirms equation (3.14).

The decomposition to the stretching ($\mathbf{D}$) and spin ($\mathbf{W}$; p. 70) is:

$$\mathbf{D} = \tfrac{1}{2}(\mathbf{L} + \mathbf{L}^T) = \mathbf{L}, \tag{10}$$

$$\mathbf{W} = \tfrac{1}{2}(\mathbf{L} - \mathbf{L}^T) = \tfrac{1}{2}t^{-1}(\mathbf{1} - \mathbf{1}) = \mathbf{0} \tag{11}$$

(the zero spin is in accordance with non-rotational character of this motion).

The determinant of the deformation gradient (p. 70) is:

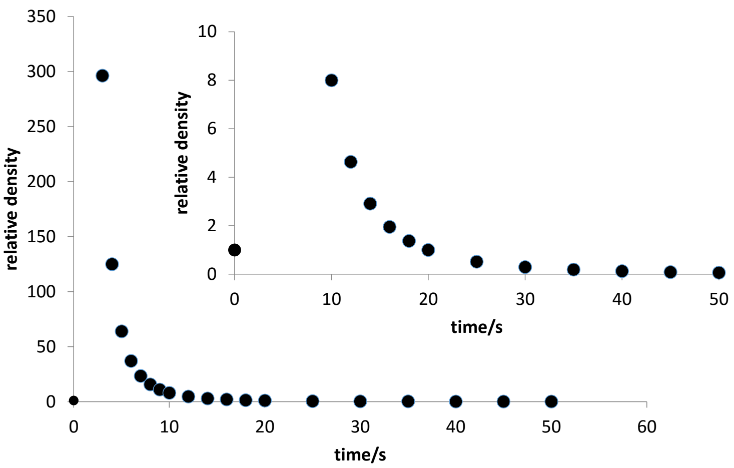
$$J \equiv |\det\mathbf{F}| = |\det(\alpha t\mathbf{1})| = |(\alpha t)^3|. \tag{12}$$

The density (field) develops in time according to the following equation (p. 87) written for the simplified case $\alpha > 0$:

$$\rho = \rho_0/J = \rho_0(\alpha t)^{-3} \tag{13}$$

where $\rho_0 > 0$ is the density in the reference configuration.

Let us investigate the specific case of $\alpha = \frac{1}{20}\,\mathrm{s}^{-1}$. The compression occurs when $\rho_0/\rho < 1$, that is, from (13), when $(t/20)^3 < 1\,\mathrm{s}$ or when $t < 20\,\mathrm{s}$. The expansion occurs when $\rho_0/\rho > 1$, that is, from (13), when $(t/20)^3 > 1\,\mathrm{s}$ or when $t > 20\,\mathrm{s}$. At $t = 20\,\mathrm{s}$, the density is equal to the referential density. Following figure shows the evolution of density within the first fifty seconds of the motion:



Another figure illustrates the shape change of a cube during the first 50 s of motion. Only its bottom face is shown and the vector **b** was selected as (1,1,0). Note the referential configuration at $t = 0\,\text{s}$.