Exercise 7 to section 3.7.[1]

Incompressible fluid flows in a rectangular slit placed in external gravitational field of the Earth. The flow is laminar, stationary, in the direction of x-axis which is the same as the direction of the gravitational force. The horizontal axis is the y-axis. The slit is formed by two parallel slabs, the distance between them (in the direction of the y-axis) is $2B$; the fluid clings to the slab walls. The slit length is L, its width W (suppose $L >> B$ and $W >> B$ to ignore additional boundary effects). The pressure at both open ends along the length L is P_0 and equal to the atmospheric pressure. The origin of Cartesian axes is located in the center of the upper opening cross-section. See also the figure below.

Integrating Navier-Stokes equation derive equation for the velocity field of the fluid, then the expressions for the maximum and mean velocity, and for the volume flow rate. The gravitational force vector magnitude $\mid \mathbf{b} \mid = g$.

Consider a slit of the following dimensions: thickness $B = 0.5$ mm, length $L = 50$ cm, width $W = 20$ cm. Water flows through the slit, its viscosity is $1\ \mathrm{g\,cm^{-1}s^{-1}}$, density $1\ \mathrm{g\,cm^{-3}}$. Calculate the volume flow rate of water and its maximum and average velocities. The gravitational acceleration (g) has the value of $9.81\ \mathrm{m\,s^{-2}}$.

Try to answer before continuing reading.

Navier-Stokes equation for the incompressible fluid is (cf. exercise 3 to section 3.7):

$$\rho\dot{\mathbf{v}} = -\mathrm{grad}P + \eta\,\mathrm{div}\,\mathrm{grad}\mathbf{v} + \rho(\mathbf{b} + \mathbf{i})$$

and in our case is simplified to

$$\mathbf{o} = -\mathrm{grad}P + \eta\,\mathrm{div}\,\mathrm{grad}\mathbf{v} + \rho\mathbf{b} \tag{1}$$

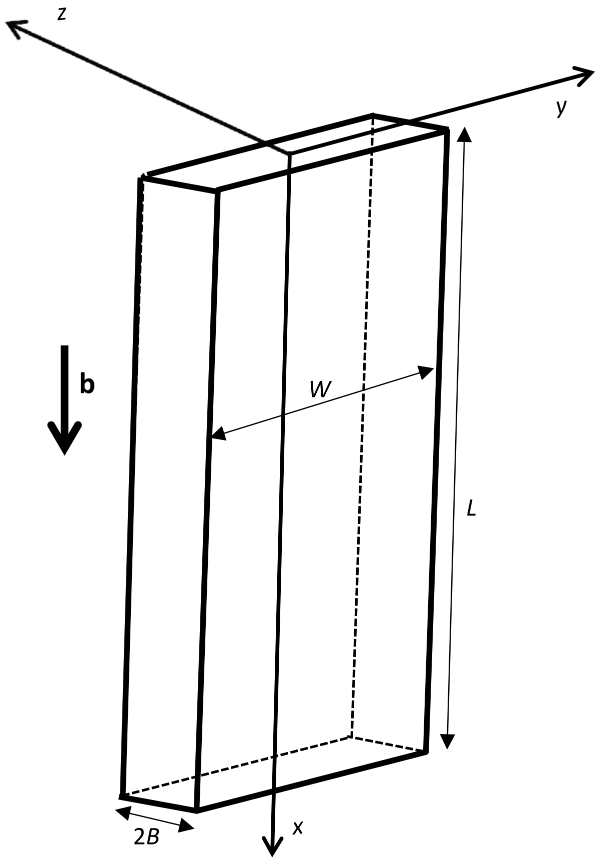
(the stationary case means that $\dot{\mathbf{v}} = \mathbf{o}$). The vector $\mathbf{b}$ has only the x-component which is equal to g.

The fluid flows along the x-axis only, thus $v^y = v^z = 0$ and $v^x \neq 0$. From the incompressibility condition ($\mathrm{div}\mathbf{v} = 0$) then follows that v^x does not depend on the x coordinate:

$$\mathrm{div}\mathbf{v} = \frac{\partial v^x}{\partial x} = 0. \tag{2}$$

The flow is planar (laminar), i.e. in the form of parallel planes, identical in all planes perpendicular to the z-axis. In other words, v^x does not depend

[1]Based on I. Samohýl: Irreversible Thermodynamics. Prague: University of Chemical Technology, 1998 (*in Czech*).



on the z coordinate, as well:

$$v^x = v^x(y). \tag{3}$$

From Navier-Stokes equation (1) then follows:

$$\frac{\partial P}{\partial x} = \eta \frac{\partial^2 v^x}{\partial y^2} + \rho g \tag{4}$$

and:

$$\frac{\partial P}{\partial y} = 0 = \frac{\partial P}{\partial z}. \tag{5}$$

Eq. (5) states that $P = P(x)$. However, the pressures at both slit openings are given to be equal, thus even $\partial P/\partial x = 0$. Eq. (4) then gives

$$\frac{\partial^2 v^x}{\partial y^2} = \frac{-\rho g}{\eta}. \tag{6}$$

First integration of (6) gives:

$$\frac{\partial v^x}{\partial y} = -\frac{\rho g}{\eta} y + C_1. \tag{7}$$

Integrating for the second time we obtain:

$$v^x = -\frac{\rho g}{2\eta} y^2 + C_1 y + C_2. \tag{8}$$

The sticking to the walls corresponds to boundary conditions $v^x = 0$ at $y = \pm B$. Consequently:

$$\begin{aligned} y = B\colon \quad 0 &= -\frac{\rho g}{2\eta} B^2 + C_1 B + C_2, \\ y = -B\colon \quad 0 &= -\frac{\rho g}{2\eta} B^2 - C_1 B + C_2. \end{aligned} \tag{9}$$

The second condition (9) gives $C_2 = B^2 \rho g/(2\eta)$ and then $C_1 = 0$ follows from the first condition.

The final result follows from substituting C_1 and C_2 into (8):

$$v^x = \frac{\rho g}{2\eta}(B^2 - y^2) = \frac{\rho g B^2}{2\eta}\left(1 - \frac{y^2}{B^2}\right). \tag{10}$$

Obviously, the maximum velocity is attained when $y = 0$, i.e., in the center of the slit:

$$v^x_{\mathrm{max}} = \frac{\rho g B^2}{2\eta}. \tag{11}$$

The average (mean) velocity is defined by integral:

$$\bar{v}^x = \frac{1}{2B} \int_{-B}^{B} v^x dy = \frac{\rho g B^2}{3\eta} \equiv \frac{2}{3} v^x_{\mathrm{max}}. \tag{12}$$

The volume flow rate is then given by:

$$Q = 2BW\bar{v}^x = \frac{2}{3}\frac{\rho g}{\eta} W B^3. \tag{13}$$

Substitution of the given numerical values into equations (12), (13), and (11) results in the average velocity of $0.8175\,\mathrm{cm\,s^{-1}}$, the volume flow rate of $1.635\,\mathrm{cm^3\,s^{-1}}$, and the maximum velocity of $1.2263\,\mathrm{cm\,s^{-1}}$.