Exercise 3 to section 3.7.[1] **Navier-Stokes equation**

Take the divergence of the stress tensor of the linear fluid, substitute it into momentum balance and derive the Navier-Stokes equation for the velocity field. Also find its simplifications – incompressible fluid, Euler (non-viscous fluid) and hydrostatics equations.

Try to answer before continuing reading.

The stress tensor is given by (3.195), substitution from (3.188) gives:

$$\mathbf{T} = -P\mathbf{1} + \zeta(\mathrm{tr}\mathbf{D})\mathbf{1} + 2\eta\mathbf{D} = -P\mathbf{1} + (\zeta - 2\eta/3)(\mathrm{tr}\mathbf{D})\mathbf{1} + 2\eta\mathbf{D}. \quad (1)$$

The divergence of (1) contains three members. The first member is $-\mathrm{div}P\mathbf{1}$ which, as easily seen, is $-\mathrm{grad}P$; remember that divergence of a tensor ($\mathbf{A}$) is a vector with components:

$$(\mathrm{div}\mathbf{A})^i = \sum_j \frac{\partial A^{ij}}{\partial x^j}. \quad (2)$$

The second member is $(\zeta - 2\eta/3)\mathrm{div}[(\mathrm{tr}\mathbf{D})\mathbf{1}]$. Equation (3.16) shows that $\mathrm{tr}\mathbf{D} = \mathrm{div}\mathbf{v}$. The term $(\mathrm{div}\mathbf{v})\mathbf{1}$ is a tensor with components:

$$[(\mathrm{div}\mathbf{v})\mathbf{1}]^{ij} = \sum_k \frac{\partial v^k}{\partial x^k}\delta^{ij} \quad (3)$$

where δ^{ij} is Kronecker's symbol. Then, $\mathrm{div}[(\mathrm{div}\mathbf{v})\mathbf{1}]$ is a vector with components, cf. (2):

$$\{\mathrm{div}[(\mathrm{div}\mathbf{v})\mathbf{1}]\}^i = \frac{\partial}{\partial x^i}\Big(\sum_k \frac{\partial v^k}{\partial x^k}\Big); \quad (4)$$

for example, its second component is as follows:

$$\frac{\partial^2 v^1}{\partial x^2 \partial x^1} + \frac{\partial^2 v^2}{\partial (x^2)^2} + \frac{\partial^2 v^3}{\partial x^2 \partial x^3}.$$

Remind that

$$\mathrm{div}\mathbf{v} = \frac{\partial v^1}{\partial x^1} + \frac{\partial v^2}{\partial x^2} + \frac{\partial v^3}{\partial x^3}$$

and thus

$$(\mathrm{grad}\,\mathrm{div}\mathbf{v})^i = \frac{\partial\,\mathrm{div}\mathbf{v}}{\partial x^i}. \quad (5)$$

[1] Based on I. Samohýl: Irreversible Thermodynamics. Prague: University of Chemical Technology, 1998 (*in Czech*).

Summarizing the development between (3) and (5) we see that the second term can be rewritten using

$$\mathrm{div}[(\mathrm{tr}\mathbf{D})\mathbf{1}] = \mathrm{grad}\,\mathrm{div}\mathbf{v}. \tag{6}$$

The third member contains div$\mathbf{D}$. The components of $\mathbf{D}$ are, cf. (3.15) and (3.14):

$$D^{ij} = \frac{1}{2}\Big(\frac{\partial v^i}{\partial x^j} + \frac{\partial v^j}{\partial x^i}\Big),$$

consequently, components of the vector div$\mathbf{D}$ are:

$$\begin{aligned}(\mathrm{div}\mathbf{D})^i &= \sum_j \frac{\partial D^{ij}}{\partial x^j} = \frac{1}{2}\sum_j \frac{\partial}{\partial x^j}\Big(\frac{\partial v^i}{\partial x^j} + \frac{\partial v^j}{\partial x^i}\Big) = \frac{1}{2}\sum_j \Big(\frac{\partial^2 v^i}{\partial (x^j)^2} + \frac{\partial^2 v^j}{\partial x^j \partial x^i}\Big)\\ &= \frac{1}{2}\Big(\frac{\partial^2 v^i}{\partial (x^1)^2} + \frac{\partial^2 v^i}{\partial (x^2)^2} + \frac{\partial^2 v^i}{\partial (x^3)^2} + \frac{\partial^2 v^1}{\partial x^1 \partial x^i} + \frac{\partial^2 v^2}{\partial x^2 \partial x^i} + \frac{\partial^2 v^3}{\partial x^3 \partial x^i}\Big)\\ &= \frac{1}{2}(\mathrm{div}\,\mathrm{grad}\mathbf{v})^i + \frac{1}{2}(\mathrm{grad}\,\mathrm{div}\mathbf{v})^i. \end{aligned} \tag{7}$$

Deriving (7), remember that grad$\mathbf{v}$ is tensor with components

$$(\mathrm{grad}\mathbf{v})^{ij} = \frac{\partial v^i}{\partial x^j}.$$

Collecting all the three divergence members gives:

$$\begin{aligned}\mathrm{div}\mathbf{T} &= -\mathrm{grad}P + (\zeta - 2\eta/3)\mathrm{grad}\,\mathrm{div}\mathbf{v} + 2\eta\frac{1}{2}(\mathrm{div}\,\mathrm{grad}\mathbf{v} + \mathrm{grad}\,\mathrm{div}\mathbf{v})\\ &= -\mathrm{grad}P + (\zeta + \eta/3)\mathrm{grad}\,\mathrm{div}\mathbf{v} + \eta\,\mathrm{div}\,\mathrm{grad}\mathbf{v}. \end{aligned} \tag{8}$$

Substituting from (8) into momentum balance (3.78), the *Navier-Stokes equation* results:

$$\rho\dot{\mathbf{v}} = -\mathrm{grad}P + (\zeta + \eta/3)\mathrm{grad}\,\mathrm{div}\mathbf{v} + \eta\,\mathrm{div}\,\mathrm{grad}\mathbf{v} + \rho(\mathbf{b} + \mathbf{i}). \tag{9}$$

In the case of *incompressible fluid* div$\mathbf{v} = 0$ (called also isochoric flow) and (9) is simplified to:

$$\rho\dot{\mathbf{v}} = -\mathrm{grad}P + \eta\,\mathrm{div}\,\mathrm{grad}\mathbf{v} + \rho(\mathbf{b} + \mathbf{i}). \tag{10}$$

The non-viscous fluid is described by zero viscosity coefficients (ζ, η); then *Euler equation* follows from (9):

$$\rho\dot{\mathbf{v}} = -\mathrm{grad}P + \rho(\mathbf{b} + \mathbf{i}). \tag{11}$$

In the special case of *hydrostatics*, i.e. non-flowing fluid ($\mathbf{v} \equiv \mathbf{o}$) we have from (9):

$$\mathrm{grad}P = \rho(\mathbf{b} + \mathbf{i}) \tag{12}$$

which is the same equation as (3.228) (why?).