Exercise 1 to section 3.7.[1] Heat conduction I

Derive the (differential) equation for the temperature change in a *fluid in rest* with no radiation. Use the results of sec. 3.7.

Try to answer before continuing reading.

The velocity is zero in a fluid in rest, $\mathbf{v} = \mathbf{o}$. Then $\mathbf{D} = \mathbf{0}$, cf. (3.15) and (3.14), and, using (3.16), $\mathrm{tr}\mathbf{D} = \mathrm{div}\mathbf{v} = 0$. Consequently, the energy balance (3.107) is transformed as follows:

$$\rho \dot{u} = -\mathrm{div}\mathbf{q}. \tag{1}$$

Substituting Fourier law (3.187) into (1) results in

$$\rho \dot{u} = \mathrm{div}(k\mathbf{g}). \tag{2}$$

The derivative of the (specific) internal energy can be expressed from (3.192):

$$\dot{u} = \frac{\partial \hat{u}}{\partial \rho}\dot{\rho} + \frac{\partial \hat{u}}{\partial T}\dot{T}. \tag{3}$$

The local mass balance (3.63) simplifies in our case to $\dot{\rho} = 0$, consequently, it follows from (3):

$$\dot{u} = c_V \frac{\partial T}{\partial t}. \tag{4}$$

Here, $c_V = \partial \hat{u}/\partial T$ is the heat heat capacity at constant volume, which is the same as at constant density, cf. (3.199). Further, we have used the fact that in the case of $\mathbf{v} = \mathbf{o}$, the temperature derivative $\dot{T} = \partial T/\partial t$, cf. (3.8).

Combining (2) and (4):

$$\rho\, c_V \frac{\partial T}{\partial t} = \mathrm{div}(k\mathbf{g})$$

and for constant k:

$$\rho\, c_V \frac{\partial T}{\partial t} = k\, \mathrm{div}\mathbf{g}.$$

Let us define the temperature conductivity

$$\alpha = \frac{k}{\rho\, c_V}$$

[1]Based on I. Samohýl: Irreversible Thermodynamics. Prague: University of Chemical Technology, 1998 (*in Czech*).

which is positive, cf. (3.232), (3.256) and text above (3.58). The final equation is then:

$$\frac{\partial T}{\partial t} = \alpha\,\mathrm{div}\mathbf{g} \equiv \alpha\,\mathrm{div}(\mathrm{grad}T).$$

One-dimensional heat conduction equation (along the coordinate x) thus is:

$$\frac{\partial T}{\partial t} = \alpha\frac{\partial^2 T}{\partial x^2}.$$