Exercise 1 to section 3.2[1]

The *rigid motion* can be expressed generally in the form $\mathbf{x} = \mathbf{\Gamma}(t)\mathbf{X} + \boldsymbol{\gamma}(t)$ where $\mathbf{\Gamma}(t)$ is the orthogonal tensor function of time (cf. exercise 3 to sect. 3.1). Find such change of frame functions $\mathbf{Q}(t)$ and $\mathbf{c}(t)$ (cf. p. 74) that the change of frame gives zero velocity $\mathbf{v}$. Try to answer before continuing reading.

The velocity transformation in the case of rigid motion is obtained substituting its above definition into (3.35):

$$\mathbf{v}^* = \mathbf{Q}\mathbf{v} + \dot{\mathbf{c}} + \dot{\mathbf{Q}}\mathbf{x} = \mathbf{Q}\mathbf{v} + \dot{\mathbf{c}} + \dot{\mathbf{Q}}(\mathbf{\Gamma}\mathbf{X} + \boldsymbol{\gamma}). \tag{1}$$

From definitions of velocity and motion, (3.7) and (3.1), respectively, it follows that $\mathbf{v} = \dot{\underline{\chi}} = \dot{\mathbf{x}}$; upon combination with the general expression for the rigid motion we have:

$$\mathbf{v} = \dot{\mathbf{\Gamma}}\mathbf{X} + \mathbf{\Gamma}\dot{\mathbf{X}} + \dot{\boldsymbol{\gamma}}. \tag{2}$$

Substituting (2) into (1) together with the requirement of zero transformed velocity results in:

$$\begin{aligned} \mathbf{0} &= \mathbf{Q}\dot{\mathbf{\Gamma}}\mathbf{X} + \mathbf{Q}\mathbf{\Gamma}\dot{\mathbf{X}} + \mathbf{Q}\dot{\boldsymbol{\gamma}} + \dot{\mathbf{c}} + \dot{\mathbf{Q}}\mathbf{\Gamma}\mathbf{X} + \dot{\mathbf{Q}}\boldsymbol{\gamma} \\ &= (\mathbf{Q}\dot{\mathbf{\Gamma}} + \dot{\mathbf{Q}}\mathbf{\Gamma})\mathbf{X} + \mathbf{Q}\mathbf{\Gamma}\dot{\mathbf{X}} + \dot{\overline{\mathbf{Q}\boldsymbol{\gamma}}} + \dot{\mathbf{c}} \\ &= \dot{\overline{\mathbf{Q}\mathbf{\Gamma}}}\mathbf{X} + \dot{\overline{\mathbf{Q}\boldsymbol{\gamma}}} + \dot{\mathbf{c}} \end{aligned} \tag{3}$$

where we used the fact that time derivative of particle $\mathbf{X}$ is zero.

Tensor $\mathbf{Q}$ is orthogonal, which, by definition, means that $\mathbf{Q}\mathbf{Q}^T = \mathbf{1}$ and from this:

$$\dot{\overline{\mathbf{Q}\mathbf{Q}^T}} = \mathbf{0}. \tag{4}$$

Comparing (4) and (3) it is seen that (3) is fulfilled when $\mathbf{\Gamma} = \mathbf{Q}^T$ and $\mathbf{c} = -\mathbf{Q}\boldsymbol{\gamma}$. Thus, the answer to this exercise is as follows:

$$\mathbf{Q} = \mathbf{\Gamma}^T, \quad \mathbf{c} = -\mathbf{\Gamma}^T\boldsymbol{\gamma}.$$

[1] Based on I. Samohýl: Irreversible Thermodynamics. Prague: University of Chemical Technology, 1998 (*in Czech*).