Exercise 1 to section 3.4

Explain the symbol $\mathbf{vT}$ which appears in equation (3.106).

Try to answer before continuing reading.

Equation (3.106) is a result of applying Gauss' theorem on (3.105). Evidently, the term in (3.105) containing $\mathbf{v}.\mathbf{Tn}$ is the source of the term $\mathbf{vT}$ in (3.106).

Gauss' theorem transforms surface integral $\int_{\partial\mathcal{V}} \mathbf{b}.\mathbf{n}\,\mathrm{d}a$ into volume integral $\int_{\mathcal{V}} \mathrm{div}\mathbf{b}\,\mathrm{d}v$. The term $\mathbf{v}.\mathbf{Tn}$ is scalar product of two vectors, one of which is $\mathbf{Tn}$. This product should be expressed in terms of scalar product with the vector $\mathbf{n}$.

The components of $\mathbf{Tn}$ are:

$$\sum_j T^{1j}n^j; \sum_j T^{2j}n^j; \sum_j T^{3j}n^j.$$

The scalar product $\mathbf{v}.\mathbf{Tn}$ is:

$$\begin{aligned}\mathbf{v}.\mathbf{Tn} =& v^1(\mathbf{Tn})^1 + v^2(\mathbf{Tn})^2 + v^3(\mathbf{Tn})^3 = \\ & v^1\sum_j T^{1j}n^j + v^2\sum_j T^{2j}n^j + v^3\sum_j T^{3j}n^j = \\ & \sum_i v^iT^{i1}n^1 + \sum_i v^iT^{i2}n^2 + \sum_i v^iT^{i3}n^3.\end{aligned}$$

We can thus write

$$\mathbf{v}.\mathbf{Tn} = (\mathbf{vT}).\mathbf{n}$$

where the i-th component of the vector $\mathbf{vT}$ is $(\mathbf{vT})^i = \sum_j v^jT^{ji}$. Gauss' theorem transformation of the appropriate term in (3.105) can be then written

$$\int_{\partial\mathcal{V}} \mathbf{v}.\mathbf{Tn}\,\mathrm{d}a \equiv \int_{\partial\mathcal{V}} (\mathbf{vT}).\mathbf{n}\,\mathrm{d}a = \int_{\mathcal{V}} \mathrm{div}(\mathbf{vT})\,\mathrm{d}v.$$